Vector Bundles and K-Theory

First installment, August 1998

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Table of Contents

Chapter 1. Vector Bundles

1. Basic Definitions and Constructions

Vector bundles are special sorts of fiber bundles with additional algebraic structure. Here is the basic definition: An *n-dimensional vector bundle* is a map $p:E\to B$ together with a real vector space structure on $p^{-1}(b)$ for each $b\in B$, such that the following local triviality condition is satisfied: There is a cover of B by open sets U_α for each of which there exists a homeomorphism $h_\alpha:p^{-1}(U_\alpha)\to U_\alpha\times\mathbb{R}^n$ taking $p^{-1}(b)$ to $\{b\}\times\mathbb{R}^n$ by a vector space isomorphism for each $b\in U_\alpha$. Such an h_α is called a *local trivialization* of the vector bundle. The space B is called the *base space*, E is the *total space*, and the vector spaces $p^{-1}(b)$ are the *fibers*. Often one abbreviates terminology by just calling the vector bundle E, letting the rest of the data be implicit. We could equally well take $\mathbb{C}$ in place of $\mathbb{R}$ as the scalar field here, obtaining the notion of a *complex vector bundle*.

If we modify the definition by dropping all references to vector spaces and replace $\mathbb{R}^n$ by an arbitrary space F, then we have the definition of a fiber bundle: a map $p:E\to B$ such that there is a cover of B by open sets U_α for each of which there exists a homeomorphism $h_\alpha:p^{-1}(U_\alpha)\to U_\alpha\times F$ taking $p^{-1}(b)$ to $\{b\}\times F$ for each $b\in U_\alpha$.

Here are some examples of vector bundles:

(1) The *product* or *trivial* bundle $E=B\times\mathbb{R}^n$ with p the projection onto the first factor.

(2) If we let E be the quotient space of $I\times\mathbb{R}$ under the identifications $(0,t)\sim(1,-t)$, then the projection $I\times\mathbb{R}\to I$ induces a map $p:E\to S^1$ which is a 1-dimensional vector bundle. Since E is homeomorphic to a Möbius band with its boundary circle deleted, we call this bundle the *Möbius bundle*.

(3) The tangent bundle of the unit sphere S^n in $\mathbb{R}^{n+1}$. Here $E=\{\,(x,v)\in S^n\times\mathbb{R}^{n+1}\mid x\perp v\,\}$, where we think of v as a tangent vector to S^n by translating it so that its tail is at the head of x, on S^n. The map $p:E\to S^n$ sends (x,v) to x. To construct local trivializations, choose any point $b\in S^n$ and let $U_b\subset S^n$ be the open hemisphere containing b and bounded by the hyperplane through the origin orthogonal to b. Define $h_b:p^{-1}(U_b)\to U_b\times p^{-1}(b)\approx U_b\times\mathbb{R}^n$ by $h_b(x,v)=(x,\pi_b(v))$ where π_b is orthogonal projection onto the tangent plane $p^{-1}(b)$. Then h_b is a local trivialization since π_b restricts to an isomorphism of $p^{-1}(x)$ onto $p^{-1}(b)$ for each $x\in U_b$.

(4) The normal bundle to S^n in $\mathbb{R}^{n+1}$, with $E\subset S^n\times\mathbb{R}^{n+1}$ consisting of pairs (x,v) such that v is perpendicular to the tangent plane to S^n at x, i.e., $v=tx$ for some $t\in\mathbb{R}$, and $p:E\to S^n$ again given by $p(x,v)=x$. As in the previous example, local trivializations $h_b:p^{-1}(U_b)\to U_b\times\mathbb{R}$ can be obtained by orthogonal projection of the fibers $p^{-1}(x)$ onto

$p^{-1}(b)$ for $x \in U_b$.

(5) The *canonical line bundle* $p: E \to \mathbb{R}\mathrm{P}^n$ where we think of $\mathbb{R}\mathrm{P}^n$ as the space of lines in $\mathbb{R}^{n+1}$ through the origin and E is the subspace of $\mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1}$ consisting of pairs (ℓ, v) with $v \in \ell$. The projection p is given by $p(\ell, v) = \ell$. Again local trivializations can be defined by orthogonal projection. We could also take $n = \infty$ and get the canonical line bundle $E \to \mathbb{R}\mathrm{P}^\infty$.

(6) The orthogonal complement $E^\perp = \{\,(\ell, v) \in \mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1} \mid v \perp \ell\,\}$ of the canonical line bundle. The projection $p: E^\perp \to \mathbb{R}\mathrm{P}^n$, $p(\ell, v) = \ell$, is a vector bundle with fibers the orthogonal subspaces $\ell^\perp$, of dimension n. Local trivializations can be obtained once more by orthogonal projection.

An *isomorphism* between vector bundles $p_1: E_1 \to B$ and $p_2: E_2 \to B$ over the same base space B is a homeomorphism $h: E_1 \to E_2$ taking each fiber $p_1^{-1}(b)$ to the corresponding fiber $p_2^{-1}(b)$ by a linear isomorphism. Thus an isomorphism preserves all the structure of a vector bundle, so isomorphic bundles are often regarded as the same. We use the notation $E_1 \approx E_2$ to indicate that E_1 and E_2 are isomorphic.

For example, the normal bundle of S^n in $\mathbb{R}^{n+1}$ is isomorphic to the product bundle $S^n \times \mathbb{R}$ by the map $(x, tx) \mapsto (x, t)$. The tangent bundle to S^1 is also isomorphic to the trivial bundle $S^1 \times \mathbb{R}$, via $(e^{i\theta}, ite^{i\theta}) \mapsto (e^{i\theta}, t)$, for $e^{i\theta} \in S^1$ and $t \in \mathbb{R}$.

As a further example, the Möbius bundle in (2) above is isomorphic to the canonical line bundle over $\mathbb{R}\mathrm{P}^1 \approx S^1$. Namely, $\mathbb{R}\mathrm{P}^1$ is swept out by a line rotating through an angle of π, so the vectors in these lines sweep out a rectangle $[0, \pi] \times \mathbb{R}$ with the two ends $\{0\} \times \mathbb{R}$ and $\{\pi\} \times \mathbb{R}$ identified. The identification is $(0, x) \sim (\pi, -x)$ since rotating a vector through an angle of π produces its negative.

The *0-section* of a vector bundle $p: E \to B$ is the union of the zero vectors in all the fibers. This is a subspace of E which projects homeomorphically onto B by p. Moreover, E deformation retracts onto its 0-section via the homotopy $f_t(v) = (1-t)v$ given by scalar multiplication of vectors $v \in E$. Thus all vector bundles over B have the same homotopy type.

One can sometimes distinguish nonisomorphic bundles by looking at the complement of the 0-section since any vector bundle isomorphism $h: E_1 \to E_2$ must take the 0-section of E_1 onto the 0-section of E_2, hence the complements of the 0-sections in E_1 and E_2 must be homeomorphic. For example, the Möbius bundle is not isomorphic to the product bundle $S^1 \times \mathbb{R}$ since the complement of the 0-section in the Möbius bundle is connected while for the product bundle the complement of the 0-section is not connected. This method for distinguishing vector bundles can also be used with more refined topological invariants such as H_n in place of H_0.

We shall denote the set of isomorphism classes of n-dimensional real vector bundles over B by $Vect^n(B)$, and its complex analogue by $Vect^n_{\mathbb{C}}(B)$. For those who worry about set theory, we are using the term "set" here in a naive sense. It follows from Theorem 1.7 that $Vect^n(B)$ and $Vect^n_{\mathbb{C}}(B)$ are indeed sets in the strict sense when B is paracompact.

For example, $Vect^1(S^1)$ contains exactly two elements, the Möbius bundle and the product bundle. This will be a rather trivial application of later theory, but it might be an interesting exercise to prove it now directly from the definitions.

Sections

A *section* of a bundle $p : E \to B$ is a map $s : B \to E$ such that $ps = 1\!\!1$, i.e., $s(b) \in p^{-1}(b)$ for all $b \in B$. We have already mentioned the 0-section, which is the section whose values are all zero. At the other extreme would be a section whose values are all nonzero. Not all vector bundles have such a nonvanishing section. Consider for example the tangent bundle to S^n. Here a section is just a tangent vector field to S^n. One of the standard first applications of homology theory is the theorem that S^n has a nonvanishing vector field iff n is odd. From this it follows that the tangent bundle of S^n is not isomorphic to the trivial bundle if n is even, since the trivial bundle obviously has a nonvanishing section, and an isomorphism between vector bundles takes nonvanishing sections to nonvanishing sections.

In fact, an n-dimensional bundle $p : E \to B$ is isomorphic to the trivial bundle iff it has n sections $s_1, \cdots, s_n$ such that $s_1(b), \cdots, s_n(b)$ are linearly independent in each fiber $p^{-1}(b)$. For if one has such sections s_i, the map $h : B \times \mathbb{R}^n \to E$ given by $h(b, t_1, \cdots, t_n) = \sum t_i s_i(b)$ is a linear isomorphism in each fiber, and is continuous, as can be verified by composing with a local trivialization $p^{-1}(U) \to U \times \mathbb{R}^n$. Hence h is an isomorphism by the following useful technical result:

Lemma 1.1. *A continuous map $h : E_1 \to E_2$ between vector bundles over the same base space B is a vector bundle isomorphism if it takes each fiber $p_1^{-1}(b)$ to the corresponding fiber $p_2^{-1}(b)$ by a linear isomorphism.*

Proof: The hypothesis implies that h is one-to-one and onto. What must be checked is that h^{-1} is continuous. This is a local question, so we may restrict to an open set $U \subset B$ over which E_1 and E_2 are trivial. Composing with local trivializations reduces to the case of an isomorphism $h : U \times \mathbb{R}^n \to U \times \mathbb{R}^n$ of the form $h(x, v) = (x, g_x(v))$. Here $g_x \in GL_n(\mathbb{R})$ depends continuously on x, i.e., the n^2 entries of g_x depend continuously on x. The inverse matrix g_x^{-1} also depends continuously on x since its entries can be expressed algebraically in terms of the entries of g_x, namely, g_x^{-1} is $1/(det g_x)$ times the classical adjoint matrix of g_x. Therefore $h^{-1}(x,v) = (x, g_x^{-1}(v))$ is continuous. $\square$

The tangent bundle to S^1 is trivial because it has the section $(x_1,x_2) \mapsto (-x_2,x_1)$ for $(x_1,x_2) \in S^1$. In terms of complex numbers, if we set $z = x_1 + ix_2$ then this section is $z \mapsto iz$ since $iz = -x_2 + ix_1$. What is the quaternion analog of this? Quaternions have the form $z = x_1 + ix_2 + jx_3 + kx_4$, and form a division algebra $\mathbb{H}$ via the multiplication rules $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, and $ik = -j$. If we identify $\mathbb{H}$ with $\mathbb{R}^4$ via the coordinates (x_1,x_2,x_3,x_4), then the unit sphere is S^3 and we can define three sections of its tangent bundle by the formulas

$$\begin{array}{lcl} z \mapsto iz & \text{or} & (x_1,x_2,x_3,x_4) \mapsto (-x_2,x_1,-x_4,x_3) \\ z \mapsto jz & \text{or} & (x_1,x_2,x_3,x_4) \mapsto (-x_3,x_4,x_1,-x_2) \\ z \mapsto kz & \text{or} & (x_1,x_2,x_3,x_4) \mapsto (-x_4,-x_4,x_2,x_1) \end{array}$$

It is easy to check that the three vectors in the last column are orthogonal to each other and to (x_1,x_2,x_3,x_4), so we have three linearly independent nonvanishing tangent vector fields on S^3, and hence the tangent bundle to S^3 is trivial.

The underlying reason why this works is that quaternion multiplication satisfies $|zw| = |z||w|$, where $|\cdot|$ is the usual norm of vectors in $\mathbb{R}^4$. Thus multiplication by a quaternion in the unit sphere S^3 is an isometry of $\mathbb{H}$. The quaternions $1,i,j,k$ form the standard orthonormal basis for $\mathbb{R}^4$, so when we multiply them by an arbitrary unit quaternion $z \in S^3$ we get a new orthonormal basis z, iz, jz, kz.

The same constructions work for the Cayley octonions, a division algebra structure on $\mathbb{R}^8$. Thinking of $\mathbb{R}^8$ as $\mathbb{H}\times\mathbb{H}$, multiplication of octonions is defined by $(z_1,z_2)(w_1,w_2) = (z_1w_1 - \overline{w}_2 z_2, z_1\overline{w}_2 + w_2 z_1)$ and satisfies the key property $|zw| = |z||w|$. This leads to the construction of seven orthogonal tangent vector fields on the unit sphere S^7, so the tangent bundle to S^7 is also trivial. As we shall show in §2.3, the only spheres with trivial tangent bundle are S^1, S^3, and S^7.

One final general remark before continuing with our next topic: Another way of characterizing the trivial bundle $E \approx B\times\mathbb{R}^n$ is to say that there is a continuous projection map $E \to \mathbb{R}^n$ which is a linear isomorphism on each fiber, since such a projection together with the bundle projection $E \to B$ gives an isomorphism $E \approx B \times \mathbb{R}^n$.

Direct Sums

As a preliminary to defining a direct sum operation on vector bundles, we make two simple observations:

(a) Given a vector bundle $p: E \to B$ and a subspace $A \subset B$, then $p: p^{-1}(A) \to A$ is clearly a vector bundle. We call this the *restriction of* E *over* A.

(b) Given vector bundles $p_1: E_1 \to B_1$ and $p_2: E_2 \to B_2$, then the product map $p_1 \times p_2: E_1 \times E_2 \to B_1 \times B_2$ is also a vector bundle, with fibers the products $p_1^{-1}(b_1) \times p_2^{-1}(b_2)$. For if $h_\alpha: p_1^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ and $h_\beta: p_2^{-1}(U_\beta) \to U_\beta \times \mathbb{R}^m$ are local trivializations for E_1 and E_2, then $h_\alpha \times h_\beta$ is a local trivialization for $E_1 \times E_2$.

Now suppose we are given two vector bundles $p_1: E_1 \to B$ and $p_2: E_2 \to B$ over the same base space B. The restriction of the product $E_1 \times E_2$ over the diagonal $B = \{(b, b) \in B \times B\}$ is then a vector bundle, called the *direct sum* $E_1 \oplus E_2 \to B$. Thus

$$E_1 \oplus E_2 = \{\, (v_1, v_2) \in E_1 \times E_2 \mid p_1(v_1) = p_2(v_2) \,\}$$

The fiber of $E_1 \oplus E_2$ over a point $b \in B$ is the product, or direct sum, of the vector spaces $p_1^{-1}(b)$ and $p_2^{-1}(b)$.

The direct sum of two trivial bundles is again a trivial bundle, clearly, but the direct sum of nontrivial bundles can also be trivial. For example, the direct sum of the tangent and normal bundles to S^n in $\mathbb{R}^{n+1}$ is the trivial bundle $S^n \times \mathbb{R}^{n+1}$ since elements of the direct sum are triples $(x, v, tx) \in S^n \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ with $x \perp v$, and the map $(x, v, tx) \mapsto (x, v + tx)$ gives an isomorphism of the direct sum bundle with $S^n \times \mathbb{R}^{n+1}$. So the tangent bundle to S^n is *stably trivial*: it becomes trivial after taking the direct sum with a trivial bundle.

As another example, the direct sum $E \oplus E^\perp$ of the canonical line bundle $E \to \mathbb{R}\mathrm{P}^n$ with its orthogonal complement, defined in example (6) above, is isomorphic to the trivial bundle $\mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1}$ via the map $(\ell, v, w) \mapsto (\ell, v + w)$ for $v \in \ell$ and $w \perp \ell$. Specializing to the case $n = 1$, both E and $E^\perp$ are isomorphic to the Möbius bundle over $\mathbb{R}\mathrm{P}^1 = S^1$, so the direct sum of the Möbius bundle with itself is the trivial bundle. This is just saying that if one takes a slab $I \times \mathbb{R}^2$ and glues the two faces $\{0\} \times \mathbb{R}^2$ and $\{1\} \times \mathbb{R}^2$ to each other via a 180 degree rotation of $\mathbb{R}^2$, the resulting vector bundle over S^1 is the same as if the glueing were by the identity map. In effect, one can gradually decrease the angle of rotation of the glueing map from 180 degrees to 0 without changing the vector bundle.

Pullback Bundles

Next we describe a procedure for using a map $f:A\to B$ to transform vector bundles over B into vector bundles over A. Given a vector bundle $p:E\to B$, let $f^*(E)=\{\,(a,v)\in A\times E\mid f(a)=p(v)\,\}$. This subspace of $A\times E$ fits into a commutative diagram

$$\begin{array}{ccc} f^*(E) & \xrightarrow{\widetilde{f}} & E \\ \big\downarrow{\scriptstyle\pi} & & \big\downarrow{\scriptstyle p} \\ A & \xrightarrow{f} & B \end{array}$$

where $\pi(a,v)=a$ and $\widetilde{f}(a,v)=v$. It is not hard to see that $\pi:f^*(E)\to A$ is also a vector bundle with fibers of the same dimension as in E. For example, we could say that $f^*(E)$ is the restriction of the vector bundle $1\!\!1\times p:A\times E\to A\times B$ over the graph of f, $\{(a,f(a))\in A\times B\}$, which we identify with A via the projection $(a,f(a))\mapsto a$. The vector bundle $f^*(E)$ is called the *pullback* or *induced* bundle.

As a trivial example, if f is the inclusion of a subspace $A\subset B$, then $f^*(E)$ is isomorphic to the restriction $p^{-1}(A)$ via the map $(a,v)\mapsto v$, since the condition $f(a)=p(v)$ just says that $v\in p^{-1}(A)$. So restriction over subspaces is a special case of pullback.

An interesting example which is small enough to be visualized completely is the pullback of the Möbius bundle $E\to S^1$ by the 2-to-1 covering map $f:S^1\to S^1$, $f(z)=z^2$. In this case the pullback $f^*(E)$ is a two-sheeted covering space of E which could be viewed as a coat of paint applied to "both sides" of the Möbius bundle. Since E has one half-twist, $f^*(E)$ has two half-twists, hence is the trivial bundle. More generally, if E_n is the pullback of the Möbius bundle by the map $z\mapsto z^n$, then E_n is trivial for n even and the Möbius bundle for n odd.

Some elementary properties of pullbacks, whose proofs are one-minute exercises in definition-chasing, are:

(i) $(fg)^*(E)\approx g^*(f^*(E))$.
(ii) If $E_1\approx E_2$ then $f^*(E_1)\approx f^*(E_2)$.
(iii) $f^*(E_1\oplus E_2)\approx f^*(E_1)\oplus f^*(E_2)$.

Now we come to our first important result:

Theorem 1.2. *Given a vector bundle $p:E\to B$ and homotopic maps $f_0,f_1:A\to B$, then the induced bundles $f_0^*(E)$ and $f_1^*(E)$ are isomorphic if A is paracompact.*

All the spaces one ordinarily encounters in algebraic and geometric topology are paracompact, for example compact Hausdorff spaces and CW complexes; see the Appendix to this chapter for more information about this.

Proof: Let $F\colon A \times I \to B$ be a homotopy from f_0 to f_1. The restrictions of $F^*(E)$ over $A \times \{0\}$ and $A \times \{1\}$ are then $f_0^*(E)$ and $f_1^*(E)$. So it will suffice to show:

$(*)$ The restrictions of a vector bundle $E \to X \times I$ over $X \times \{0\}$ and $X \times \{1\}$ are isomorphic if X is paracompact.

To prove $(*)$ we need two preliminary facts:

(1) A vector bundle $p\colon E \to X \times [a,b]$ is trivial if its restrictions over $X \times [a,c]$ and $X \times [c,b]$ are trivial for some $c \in (a,b)$. To see this, let these restrictions be $E_1 = p^{-1}(X \times [a,c])$ and $E_2 = p^{-1}(X \times [c,b])$, and let $h_1\colon E_1 \to X \times [a,c] \times \mathbb{R}^n$ and $h_2\colon E_2 \to X \times [c,b] \times \mathbb{R}^n$ be isomorphisms. These isomorphisms may not agree on $p^{-1}(X \times \{c\})$, but they will after we replace h_2 by its composition with the isomorphism $X \times [c,b] \times \mathbb{R}^n \to X \times [c,b] \times \mathbb{R}^n$ which on each slice $X \times \{x\} \times \mathbb{R}^n$ is given by $h_1 h_2^{-1}\colon X \times \{c\} \times \mathbb{R}^n \to X \times \{c\} \times \mathbb{R}^n$. Once h_1 and h_2 agree on $E_1 \cap E_2$, they define a trivialization of E.

(2) For a vector bundle $p\colon E \to X \times I$, there exists an open cover $\{U_\alpha\}$ of X so that each restriction $p^{-1}(U_\alpha \times I) \to U_\alpha \times I$ is trivial. This is because for each $x \in X$ we can find open neighborhoods $U_{x,1}, \cdots, U_{x,k}$ in X and a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ of $[0,1]$ such that the bundle is trivial over $U_{x,i} \times [t_{i-1}, t_i]$, using compactness of $[0,1]$. Then by (1) the bundle is trivial over $\bigcap_i U_{x,i} \times I$.

Now we prove $(*)$. By (2), we can choose an open cover $\{U_\alpha\}$ of X so that E is trivial over each $U_\alpha \times I$. Apply Lemma 1.14 in the Appendix to this chapter to produce a countable cover $\{V_k\}_{k \ge 1}$ of X and a partition of unity $\{\varphi_k\}$ with φ_k supported in V_k, such that each V_k is a disjoint union of open sets each contained in some U_α. This means that E is trivial over each $V_k \times I$.

For $k \ge 0$, let $\psi_k = \varphi_1 + \cdots + \varphi_k$, with $\psi_0 = 0$. Let X_k be the graph of ψ_k, $X_k = \{\,(x, \psi_k(x)) \in X \times I\,\}$, and let $p_k\colon E_k \to X_k$ be the restriction of the bundle E over X_k. Choosing a trivialization of E over $V_k \times I$, the natural projection homeomorphism $X_k \to X_{k-1}$ lifts to an isomorphism $h_k\colon E_k \to E_{k-1}$ which is the identity outside $p_k^{-1}(V_k)$. The infinite composition $h = h_1 h_2 \cdots$ is then a well-defined isomorphism from the restriction of E over $X \times \{0\}$ to the restriction over $X \times \{1\}$ since near each point $x \in X$ only finitely many φ_i's are nonzero, so for large enough k, $h_k = \mathbb{1}$ over a neighborhood of x. □

Corollary 1.3. *A homotopy equivalence $f\colon A \to B$ of paracompact spaces induces a bijection $f^*\colon Vect^n(B) \to Vect^n(A)$. In particular, every vector bundle over a contractible paracompact base is trivial.*

Proof: If g is a homotopy inverse of f then $f^* g^* = \mathbb{1}^* = \mathbb{1}$ and $g^* f^* = \mathbb{1}^* = \mathbb{1}$. □

We might remark that Theorem 1.2 holds for fiber bundles as well as vector bundles, with the same proof.

Inner Products

An *inner product* on a vector bundle $p: E \to B$ is a map $\langle , \rangle : E \oplus E \to \mathbb{R}$ which restricts in each fiber to an inner product, i.e., a positive definite symmetric bilinear form.

Proposition 1.4. *An inner product exists for a vector bundle $p: E \to B$ if B is paracompact.*

Proof: An inner product for $p: E \to B$ can be constructed by first using local trivializations $h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$, to pull back the standard inner product in $\mathbb{R}^n$ to an inner product $\langle \cdot, \cdot \rangle_\alpha$ on $p^{-1}(U_\alpha)$, then setting $\langle v, w \rangle = \sum_\beta \varphi_\beta p(v) \langle v, w \rangle_{\alpha(\beta)}$ where $\{\varphi_\beta\}$ is a partition of unity with the support of φ_β contained in $U_{\alpha(\beta)}$. $\square$

In the case of complex vector bundles one can construct in the same way Hermitian inner products.

Having an inner product on a vector bundle E, lengths of vectors are defined, and so we can speak of the associated unit sphere and unit disk bundles $S(E) \to B$ and $D(E) \to B$, which are fiber bundles with fibers spheres and disks. It is possible to describe the sphere bundle $S(E)$ and the disk bundle $D(E)$ without reference to an inner product, Namely, $S(E)$ is the quotient of the complement of the zero section in E obtained by identifying each nonzero vector with all positive scalar multiples of itself. And then $D(E)$ is the mapping cylinder of the projection $S(E) \to B$.

For the canonical line bundle $E \to \mathbb{R}\mathrm{P}^n$ the unit sphere bundle $S(E)$ is the space of unit vectors in lines through the origin in $\mathbb{R}^{n+1}$. Since each unit vector uniquely determines the line containing it, $S(E)$ is the same as the space of unit vectors in $\mathbb{R}^{n+1}$, i.e., S^n. For the trivial bundle $\mathbb{R}\mathrm{P}^n \times \mathbb{R}$ the unit sphere bundle is $\mathbb{R}\mathrm{P}^n \times S^0$, so the canonical line bundle is nontrivial.

Similarly, in the complex case the canonical line bundle $E \to \mathbb{C}\mathrm{P}^n$ has $S(E)$ equal to the unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$. So this bundle is not the same as the trivial bundle which has unit sphere bundle $\mathbb{C}\mathrm{P}^n \times S^1$, and these two unit sphere bundles are not homeomorphic, having different fundamental groups for example, assuming $n > 0$.

Subbundles

A *vector subbundle* of a vector bundle $p: E \to B$ has the natural definition: a subspace $E_0 \subset E$ intersecting each fiber of E in a vector subspace, such that the restriction $p: E_0 \to B$ is a vector bundle.

Proposition 1.5. *If $E \to B$ is a vector bundle over a paracompact base B and $E_0 \subset E$ is a vector subbundle, then there is a vector subbundle $E_0^\perp \subset E$ such that $E_0 \oplus E_0^\perp \approx E$.*

Proof: With respect to a chosen inner product on E, let $E_0^\perp$ be the set of vectors in fibers of E which are orthogonal to all vectors in E_0. We claim that $E_0^\perp$ is a vector bundle. If this is so, then we obviously have $E_0 \oplus E_0^\perp \approx E$.

To see that $E_0^\perp$ satisfies the local triviality condition for a vector bundle, note first that we may assume E is the product $B \times \mathbb{R}^n$ since the question is local in B. Since E_0 is a vector bundle, of dimension m say, it has m independent local sections $b \mapsto (b, s_i(b))$ near each point $b_0 \in B$. We may enlarge this set of m independent local sections of E_0 to a set of n independent local sections $b \mapsto (b, s_i(b))$ of E by choosing $s_{m+1}, \cdots, s_n$ first in the fiber $p^{-1}(b_0)$, then taking the same vectors for all nearby fibers, since if $s_1, \cdots, s_m, s_{m+1}, \cdots, s_n$ are independent at b_0, they will remain independent for nearby b by continuity of the determinant function. Apply the Gram-Schmidt orthogonalization process to $s_1, \cdots, s_m, s_{m+1}, \cdots, s_n$ in each fiber, using the given inner product, to obtain new maps s_i'. The explicit formulas for the Gram-Schmidt process show the s_i''s are continuous. The maps s_i' allow us to define a local trivialization $h : p^{-1}(U) \to U \times \mathbb{R}^n$ with $h(b, s_i'(b))$ equal to the i^{th} standard basis vector of $\mathbb{R}^n$. Then h carries E_0 to $U \times \mathbb{R}^m$ and $E_0^\perp$ to $U \times \mathbb{R}^{n-m}$, so $h \,|\, E_0^\perp$ is a local trivialization of $E_0^\perp$. $\square$

Tensor Products

In addition to direct sum, a number of other algebraic constructions with vector spaces can be extended to vector bundles. One which is particularly important for K-theory is tensor product. For vector bundles $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$, let $E_1 \otimes E_2$, as a set, be the disjoint union of the vector spaces $p_1^{-1}(x) \otimes p_2^{-1}(x)$ for $x \in B$. The topology on this set is defined in the following way. Choose isomorphisms $h_i : p_i^{-1}(U) \to U \times \mathbb{R}^{n_i}$ for each open set $U \subset B$ over which E_1 and E_2 are trivial. Then a topology $\mathcal{T}_U$ on the set $p_1^{-1}(U) \otimes p_2^{-1}(U)$ is defined by letting the fiberwise tensor product map $h_1 \otimes h_2 : p_1^{-1}(U) \otimes p_2^{-1}(U) \to U \times (\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2})$ be a homeomorphism. The topology $\mathcal{T}_U$ is independent of the choice of the h_i's since any other choices are obtained by composing with isomorphisms of $U \times \mathbb{R}^{n_i}$ of the form $(x, v) \mapsto (x, g_i(x)(v))$ for continuous maps $g_i : U \to GL_{n_i}(\mathbb{R})$, hence $h_1 \otimes h_2$ changes by composing with analogous isomorphisms of $U \times (\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2})$ whose second coordinates $g_1 \otimes g_2$ are continuous maps $U \to GL_{n_1 n_2}(\mathbb{R})$, since the entries of the matrices $g_1(x) \otimes g_2(x)$ are the products of the entries of $g_1(x)$ and $g_2(x)$. When we replace U by an open subset V, the topology on $p_1^{-1}(V) \otimes p_2^{-1}(V)$ induced by $\mathcal{T}_U$ is the same as the topology $\mathcal{T}_V$ since local trivializations over U restrict to local trivializations over V. Hence we get a well-defined topology on $E_1 \otimes E_2$ making it a vector bundle over B.

Another way to look at this construction is the following. In general, if we are given a vector bundle $p: E \to B$ and an open cover $\{U_\alpha\}$ of B with local trivializations $h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$, then we can reconstruct E as the quotient space of the disjoint union $\coprod_\alpha (U_\alpha \times \mathbb{R}^n)$ obtained by identifying $(x, v) \in U_\alpha \times \mathbb{R}^n$ with $h_\beta h_\alpha^{-1}(x, v) \in U_\beta \times \mathbb{R}^n$ whenever $x \in U_\alpha \cap U_\beta$. The functions $h_\beta h_\alpha^{-1}$ can be viewed as maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$. Now suppose we have two vector bundles $E_1 \to B$ and $E_2 \to B$. We can choose an open cover $\{U_\alpha\}$ over which both bundles are trivial, and so obtain functions $g^1_{\alpha\beta}, g^2_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_{n_i}(\mathbb{R})$ for E_1 and E_2. Then the functions "$g_{\alpha\beta}$" for the bundle $E_1 \otimes E_2$ are the tensor product functions $g^1_{\alpha\beta} \otimes g^2_{\alpha\beta}$ assigning to each $x \in U_\alpha \cap U_\beta$ the tensor product of the two matrices $g^1_{\alpha\beta}(x)$ and $g^2_{\alpha\beta}(x)$.

It is routine to verify that the tensor product operation for vector bundles over a fixed base space is commutative, associative, and has an identity element, the trivial line bundle. It is also distributive with respect to direct sum.

If we restrict attention to line bundles, then $Vect^1(B)$ is in fact an abelian group with respect to the tensor product operation, at least when B is paracompact. To show that inverses exist, let $E \to B$ be a line bundle. After choosing an inner product for E, we may rescale local trivializations h_α to be isometries, taking vectors in fibers of E to vectors in $\mathbb{R}^1$ of the same length. Then all the values of the glueing functions $g_{\alpha\beta}$ are ± 1, being isometries of $\mathbb{R}$. The glueing functions for $E \otimes E$ are the squares of these $g_{\alpha\beta}$'s, hence are identically 1, so $E \otimes E$ is the trivial line bundle. Thus each element of $Vect^1(B)$ is its own inverse. As we shall see in §3.1, the group $Vect^1(B)$ is isomorphic to $H^1(B; \mathbb{Z}_2)$ when B is homotopy equivalent to a CW complex.

These tensor product constructions work equally well for complex vector bundles. The only difference is in constructing inverses in $Vect^1_{\mathbb{C}}(B)$. Here after rescaling the glueing functions $g_{\alpha\beta}$ for a complex line bundle E, the values are complex numbers of norm 1, not necessarily ± 1, so we cannot expect $E \otimes E$ to be trivial. Instead we make use of another general construction for complex vector bundles $E \to B$, the notion of the *conjugate bundle* $\overline{E} \to B$. As a topological space, $\overline{E}$ is the same as E, but the vector space structure in the fibers is modified by redefining scalar multiplication by $\lambda(v) = \overline{\lambda} v$ where the right side of this equation means scalar multiplication in E and the left side scalar multiplication in $\overline{E}$. This means that local trivializations for $\overline{E}$ are obtained from local trivializations for E by composing with the coordinatewise conjugation map $\mathbb{C}^n \to \mathbb{C}^n$ in each fiber. The effect on the glueing maps $g_{\alpha\beta}$ is to replace them by their complex conjugates too. Specializing to line bundles, we then have $E \otimes \overline{E}$ isomorphic to the trivial line bundle since its glueing maps have values $z\bar{z} = 1$ for z a unit complex number. In §3.1 we will show that the group $Vect^1_{\mathbb{C}}(B)$ is isomorphic to $H^2(B;\mathbb{Z})$ when B is homotopy equivalent to a CW complex.

Besides tensor product of vector bundles, another construction useful in K-theory is the exterior power $\lambda^k(E)$ of a vector bundle E. Recall from linear algebra that the exterior power $\lambda^k(V)$ of a vector space V is the quotient of the k-fold tensor product $V\otimes\cdots\otimes V$ by the subspace generated by vectors of the form $v_1\otimes\cdots\otimes v_k - sgn(\sigma)v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(k)}$ where σ is a permutation of the subscripts and $sgn(\sigma) = \pm 1$ is its sign, $+1$ for an even permutation and -1 for an odd permutation. If V has dimension n then $\lambda^k(V)$ has dimension $\binom{n}{k}$. Now to define $\lambda^k(E)$ for a vector bundle $p: E \to B$ the procedure follows closely what we did for tensor product. We first form the disjoint union of the exterior powers $\lambda^k(p^{-1}(x))$ of all the fibers $p^{-1}(x)$, then we define a topology on this set via local trivializations. The key fact about tensor product which we needed before was that the tensor product $\varphi\otimes\psi$ of linear transformations φ and ψ depends continuously on φ and ψ. For exterior powers the analogous fact is that a linear map $\varphi:\mathbb{R}^n \to \mathbb{R}^n$ induces a linear map $\lambda^k(\varphi):\lambda^k(\mathbb{R}^n) \to \lambda^k(\mathbb{R}^n)$ which depends continuously on φ. This holds since $\lambda^k(\varphi)$ is a quotient map of the k-fold tensor product of φ with itself.

Associated Bundles

There are a number of geometric operations on vector spaces which can also be performed on vector bundles. As an example we have already seen, consider the operation of taking the unit sphere or unit disk in a vector space with an inner product. Given a vector bundle $E \to B$ with an inner product, we can then perform the operation in each fiber, producing the sphere bundle $S(E) \to B$ and the disk bundle $D(E) \to B$. Here are some more examples:

(1) Associated to a vector bundle $E \to B$ is the *projective bundle* $P(E) \to B$, where $P(E)$ is the space of all lines through the origin in all the fibers of E. We topologize $P(E)$ as the quotient of the sphere bundle $S(E)$ obtained by factor out scalar multiplication in each fiber. Over a neighborhood U in B where E is a product $U\times\mathbb{R}^n$, this quotient is $U\times\mathbb{R}\mathrm{P}^{n-1}$, so $P(E)$ is a fiber bundle over B with fiber $\mathbb{R}\mathrm{P}^{n-1}$, with respect to the projection $P(E) \to B$ which sends each line in the fiber of E over a point $b\in B$ to b. We could just as well start with an n-dimensional vector bundle over $\mathbb{C}$, and then $P(E)$ would have fibers $\mathbb{C}\mathrm{P}^{n-1}$.

(2) For an n-dimensional vector bundle $E \to B$, the associated *flag bundle* $F(E) \to B$ has total space $F(E)$ the subspace of the n-fold product of $P(E)$ with itself consisting

of n-tuples of orthogonal lines in fibers of E. The fiber of $F(E)$ is thus the flag manifold $F(\mathbb{R}^n)$ consisting of n-tuples of orthogonal lines through the origin in $\mathbb{R}^n$. Local triviality follows as in the preceding example. More generally, for any $k \leq n$ one could take k-tuples of orthogonal lines in fibers of E and get a bundle $F_k(E) \to B$.

(3) As a refinement of the last example, one could form the *Stiefel bundle* $V_k(E) \to B$, where points of $V_k(E)$ are k-tuples of orthogonal unit vectors in fibers of E, so $V_k(E)$ is a subspace of the product of k copies of $S(E)$. The fiber of $V_k(E)$ is the Stiefel manifold $V_k(\mathbb{R}^n)$ of orthonormal k-frames in $\mathbb{R}^n$.

(4) Generalizing $P(E)$, there is the *Grassmann bundle* $G_k(E) \to B$ of k-dimensional linear subspaces of fibers of E. This is the quotient space of $V_k(E)$ obtained by identifying two k-frames if they span the same subspace of a fiber. The fiber of $G_k(E)$ is the Grassmann manifold $G_k(\mathbb{R}^n)$ of k-planes through the origin in $\mathbb{R}^n$.

Some of these associated fiber bundles have natural vector bundles lying over them. For example, there is a canonical line bundle $L \to P(E)$ where $L = \{\, (\ell, v) \in P(E) \times E \mid v \in \ell \,\}$. Similarly, over the flag bundle $F(E)$ there are n line bundles L_i consisting of all vectors in the i^{th} line of an n-tuple of orthogonal lines in fibers of E. The direct sum $L_1 \oplus \cdots \oplus L_n$ is then equal to the pullback of E over $F(E)$,

$$\begin{array}{ccc} L_1 \oplus \cdots \oplus L_n & \longrightarrow & E \\ \downarrow & & \downarrow \\ F(E) & \longrightarrow & B \end{array}$$

since a point in the pullback consists of an n-tuple of lines $\ell_1 \perp \cdots \perp \ell_n$ in a fiber of E together with a vector v in this fiber, and v can be expressed uniquely as a sum $v = v_1 + \cdots + v_n$ with $v_i \in \ell_i$. Thus we see an interesting fact: *For every vector bundle there is a pullback which splits as a direct sum of line bundles.* This observation plays a role in the so-called "splitting principle," as we shall see in Corollary 2.23 and Proposition 3.2.

2. Classifying Vector Bundles

In this section we give two homotopy-theoretic descriptions of $Vect^n(X)$. The first works for arbitrary paracompact X, while the second is restricted to the case that X is a suspension. This allows the explicit calculation of a few simple examples, such as $X = S^n$ for $n \le 4$.

The Universal Bundle

Our aim is to construct a certain n-dimensional vector bundle $E_n \to G_n$ with the property that all n-dimensional bundles over paracompact base spaces are obtainable as pullbacks of this one bundle. When $n = 1$ this bundle will be just the canonical line bundle over $\mathbb{R}\mathrm{P}^\infty$, defined earlier. The generalization to $n > 1$ will consist in replacing $\mathbb{R}\mathrm{P}^\infty$, the space of 1-dimensional linear subspaces of $\mathbb{R}^\infty$, by a space $G_n(\mathbb{R}^\infty)$ of n-dimensional linear subspaces of $\mathbb{R}^\infty$. Just as $\mathbb{R}\mathrm{P}^\infty$ is the union of finite-dimensional projective spaces, so $G_n(\mathbb{R}^\infty)$ will be the union of the so-called *Grassmann manifolds* $G_n(\mathbb{R}^k)$ of n-dimensional linear subspaces of $\mathbb{R}^k$.

The definition of $G_n(\mathbb{R}^k)$ as a set is clear: all the n-dimensional planes in $\mathbb{R}^k$ passing through the origin. To define the topology on $G_n(\mathbb{R}^k)$ we first define the *Stiefel manifold* $V_n(\mathbb{R}^k)$ to be the space of orthonormal n-frames in $\mathbb{R}^k$, i.e., n-tuples of orthonormal vectors in $\mathbb{R}^k$. This is a subspace of the product of n copies of the unit sphere S^{k-1}, namely, the subspace of orthogonal n-tuples. It is a closed subspace since orthogonality is expressed by an algebraic equation. Hence $V_n(\mathbb{R}^k)$ is compact since the product of spheres is compact. There is a natural surjection $V_n(\mathbb{R}^k) \to G_n(\mathbb{R}^k)$ sending an n-frame to the subspace it spans, and $G_n(\mathbb{R}^k)$ is topologized by giving it the quotient topology with respect to this surjection. So $G_n(\mathbb{R}^k)$ is compact as well.

Define $E_n(\mathbb{R}^k) = \{\, (\ell, v) \in G_n(\mathbb{R}^k) \times \mathbb{R}^k \mid v \in \ell \,\}$. The inclusions $\mathbb{R}^k \subset \mathbb{R}^{k+1} \subset \cdots$ give inclusions $G_n(\mathbb{R}^k) \subset G_n(\mathbb{R}^{k+1}) \subset \cdots$ and $E_n(\mathbb{R}^k) \subset E_n(\mathbb{R}^{k+1}) \subset \cdots$. We set $G_n = G_n(\mathbb{R}^\infty) = \bigcup_k G_n(\mathbb{R}^k)$ and $E_n = E_n(\mathbb{R}^\infty) = \bigcup_k E_n(\mathbb{R}^k)$ with the weak, or direct limit, topologies. Thus a set in $G_n(\mathbb{R}^\infty)$ is open iff it intersects each $G_n(\mathbb{R}^k)$ in an open set, and similarly for $E_n(\mathbb{R}^\infty)$.

Lemma 1.6. *The projection $p : E_n(\mathbb{R}^k) \to G_n(\mathbb{R}^k)$, $p(\ell, v) = \ell$, is a vector bundle for each $k \le \infty$.*

Proof: First suppose k is finite. For $\ell \in G_n(\mathbb{R}^k)$, let $\pi_\ell : \mathbb{R}^k \to \ell$ be orthogonal projection and let $U_\ell = \{\, \ell' \in G_n(\mathbb{R}^k) \mid \pi_\ell(\ell') \text{ has dimension } n \,\}$. Thus $\ell \in U_\ell$. We will show that U_ℓ is open in $G_n(\mathbb{R}^k)$ and that the map $h : p^{-1}(U_\ell) \to U_\ell \times \ell \approx U_\ell \times \mathbb{R}^n$ defined by $h(\ell', v) = (\ell', \pi_\ell(v))$ is a local trivialization of $E_n(\mathbb{R}^k)$. It is clear that h is a bijection which is a linear isomorphism on each fiber, so it only needs to be checked that h and h^{-1} are continuous.

First we verify that U_ℓ is open. This is equivalent to saying that its preimage in $V_n(\mathbb{R}^k)$ is open. This preimage consists of orthonormal frames $v_1, \cdots, v_n$ such that $\pi_\ell(v_1), \cdots, \pi_\ell(v_n)$ are independent. Let A be the matrix of π_ℓ with respect to the standard basis in the domain $\mathbb{R}^k$ and any fixed basis in the range ℓ. The condition on $v_1, \cdots, v_n$ is then that the $n \times n$ matrix with columns $Av_1, \cdots, Av_n$ has nonzero determinant. Since the value of this determinant is obviously a continuous function of $v_1, \cdots, v_n$, it follows that the frames $v_1, \cdots, v_n$ yielding a nonzero determinant form an open set in $V_n(\mathbb{R}^k)$.

Now we check the continuity of h and h^{-1}. For $\ell' \in U_\ell$, let $L_{\ell'} : \mathbb{R}^k \to \mathbb{R}^k$ be the linear map restricting to π_ℓ on ℓ' and the identity on $\ell^\perp$. Thus $L_{\ell'}$ is invertible. We claim that $L_{\ell'}$, regarded as a $k \times k$ matrix, depends continuously on ℓ'. Namely, we can write $L_{\ell'}$ as a product AB^{-1} where B sends the standard basis to $v_1, \cdots, v_n, v_{n+1}, \cdots, v_k$ with $v_1, \cdots, v_n$ an orthonormal basis for ℓ' and $v_{n+1}, \cdots, v_k$ a fixed basis for $\ell^\perp$, and A sends the standard basis to $\pi_\ell(v_1), \cdots, \pi_\ell(v_n), v_{n+1}, \cdots, v_k$. Since matrix multiplication and matrix inversion are continuous operations (think of the 'classical adjoint' formula for the inverse of a matrix), it follows that the product $L_{\ell'} = AB^{-1}$ depends continuously on $v_1, \cdots, v_n$. But since $L_{\ell'}$ depends only on ℓ', not on the basis for ℓ', it follows that $L_{\ell'}$ depends continuously on ℓ' since $G_n(\mathbb{R}^k)$ has the quotient topology from $V_n(\mathbb{R}^k)$. Since we have $h(\ell', v) = (\ell', \pi_\ell(v)) = (\ell', L_{\ell'}(v))$, we see that h is continuous. Similarly, $h^{-1}(\ell', w) = (\ell', L_{\ell'}^{-1}(w))$ and $L_{\ell'}^{-1}$ depends continuously on ℓ', matrix inversion being continuous, so h^{-1} is continuous.

This finishes the proof for finite k. When $k = \infty$ one takes U_ℓ to be the union of the U_ℓ's for increasing k, and the local trivializations h constructed above for finite k fit together to give a local trivialization over this U_ℓ, continuity being automatic since we use the weak topology. □

Let $[X, Y]$ denote the set of homotopy classes of maps $f : X \to Y$.

Theorem 1.7. *For paracompact* X, *the map* $[X, G_n] \to Vect^n(X)$, $[f] \mapsto f^*(E_n)$, *is a bijection.*

Thus, vector bundles over a fixed base space are classified by homotopy classes of maps into G_n. Because of this, G_n is called the *classifying space* for n-dimensional vector bundles and $E_n \to G_n$ is called the *universal bundle*.

Proof: The key observation is the following: For an n-dimensional vector bundle $p : E \to X$, an isomorphism $E \approx f^*(E_n)$ is equivalent to a map $g : E \to \mathbb{R}^\infty$ which is a linear injection on each fiber. To see this, suppose first that we have a map $f: X \to G_n$ and an isomorphism $E \approx f^*(E_n)$. Then we have a commutative diagram

$$\begin{tikzcd} E \arrow[dr, "p"'] & \approx f^*(E_n) \arrow[r, "\widetilde{f}"] \arrow[d] & E_n \arrow[r, "\pi"] \arrow[d] & \mathbb{R}^\infty \\ & X \arrow[r, "f"] & G_n & \end{tikzcd}$$

where $\pi(\ell, v) = v$. The composition across the top row is a map $g: E \to \mathbb{R}^\infty$ which is a linear injection on each fiber, since both $\widetilde{f}$ and π have this property. Conversely, having $g: E \to \mathbb{R}^\infty$ which is a linear injection on each fiber, define $f: X \to G_n$ by letting $f(x)$ be the n-plane $g(p^{-1}(x))$. This clearly yields a commutative diagram as above.

To show surjectivity of the map $[X, G_n] \longrightarrow Vect^n(X)$, suppose $p: E \to X$ is an n-dimensional bundle. Let $\{U_\alpha\}$ be an open cover of X such that E is trivial over each U_α. By Lemma 1.16 in the Appendix to this chapter there is a countable open cover $\{U_i\}$ of X such that E is trivial over each U_i, and there is a partition of unity $\{\varphi_i\}$ with φ_i supported in U_i. Let $g_i: p^{-1}(U_i) \to \mathbb{R}^n$ be the composition of a trivialization $p^{-1}(U_i) \to U_i \times \mathbb{R}^n$ with projection onto $\mathbb{R}^n$. The scalar multiple $(\varphi_i p) g_i$ extends to a map $E \to \mathbb{R}^n$ which is zero outside $p^{-1}(U_i)$. Since near each point of X only finitely many φ_i's are nonzero, and at least one φ_i is nonzero, these extended $(\varphi_i p) g_i$'s are the coordinates of a map $g: E \to (\mathbb{R}^n)^\infty = \mathbb{R}^\infty$ which is a linear injection on each fiber.

For injectivity, if we have isomorphisms $E \approx f_0^*(E_n)$ and $E \approx f_1^*(E_n)$ for two maps $f_0, f_1: X \to G_n$, then these give maps $g_0, g_1: E \to \mathbb{R}^\infty$ which are linear injections on fibers, as in the first paragraph of the proof. We claim g_0 and g_1 are homotopic through maps g_t which are linear injections on fibers. If this is so, then f_0 and f_1 will be homotopic via $f_t(x) = g_t(p^{-1}(x))$.

The first step in constructing a homotopy g_t is to compose g_0 with the homotopy of $\mathbb{R}^\infty$ defined by the linear maps $L_t(x_1, x_2, \cdots) = (1-t)(x_1, x_2, \cdots) + t(x_1, 0, x_2, 0, \cdots)$. The kernel of L_t is readily computed to be 0, so L_t is injective. The effect of this homotopy on g_0 is to move it into the odd-numbered coordinates. Similarly we can homotope g_1 into the even-numbered coordinates. Still calling the new g's g_0 and g_1, let $g_t = (1-t)g_0 + t g_1$. This is clearly linear and injective on fibers since g_0 and g_1 are. $\square$

Usually $[X, G_n]$ is too difficult to compute explicitly, so this theorem is of limited use as a tool for explicitly classifying vector bundles over a given base space. Its importance lies more in the theoretical direction. Among other things, it can reduce the proof of a general statement to the special case of the universal bundle. For example, it is easy to deduce that vector bundles over a paracompact base have inner products, since the bundle

$E_n \to G_n$ has an obvious inner product obtained by restricting the standard inner product in $\mathbb{R}^\infty$ to each n-plane, and this induces an inner product on every pullback $f^*(E_n)$.

The proof of the following result will provide another illustration of this principle of the 'universal example:'

Proposition 1.8. *For each vector bundle $E \to X$ over a compact Hausdorff base space X there exists a vector bundle $E' \to X$ such that $E \oplus E'$ is the trivial bundle.*

This can fail when X is noncompact. An example is the canonical line bundle over $\mathbb{R}\mathrm{P}^\infty$, as we shall see in §3.1. It is an exercise at the end of the chapter to show that the proposition holds also whenever X is a finite-dimensional CW complex, even a noncompact one, with infinitely many cells.

Proof: First we show this holds for $E_n(\mathbb{R}^k)$. In this case the bundle with the desired property will be $E_n^\perp(\mathbb{R}^k) = \{\,(\ell, v) \in G_n(\mathbb{R}^k) \times \mathbb{R}^k \mid v \perp \ell\,\}$. Identifying $G_n(\mathbb{R}^k)$ with $G_{k-n}(\mathbb{R}^k)$ by taking orthogonal complements, the projection $E_n^\perp(\mathbb{R}^k) \to G_n(\mathbb{R}^k)$, $(\ell, v) \mapsto \ell$, is just the bundle $E_{k-n}(\mathbb{R}^k) \to G_{k-n}(\mathbb{R}^k)$. The sum $E_n(\mathbb{R}^k) \oplus E_n^\perp(\mathbb{R}^k)$ is isomorphic to the trivial bundle $G_n(\mathbb{R}^k) \times \mathbb{R}^k$ via the map $(\ell, v, w) \mapsto (\ell, v + w)$.

Now for the general case. Let $f : X \to G_n$ pull the universal bundle E_n back to the given bundle $E \to X$. The space G_n is the union of the subspaces $G_n(\mathbb{R}^k)$ for $k \geq 1$, with the weak topology, so the following lemma implies that the compact set $f(X)$ must lie in $G_n(\mathbb{R}^k)$ for some k. (The hypothesis that points in $G_n(\mathbb{R}^k)$ are closed holds since the set of orthonormal n-frames in a fixed n-plane is closed in $V_n(\mathbb{R}^k)$.) Then f pulls the trivial bundle $E_n(\mathbb{R}^k) \oplus E_n^\perp(\mathbb{R}^k)$ back to $E \oplus f^*(E_n^\perp(\mathbb{R}^k))$, which is therefore also trivial. $\square$

Lemma 1.9. *If X is the union of a sequence of subspaces $X_1 \subset X_2 \subset \cdots$ with the weak topology, and points are closed subspaces in each X_i, then for each compact set $C \subset X$ there is an X_i which contains C.*

Proof: If the conclusion is false, then for each i there is a point $x_i \in C$ not in X_i. Let $S = \{x_1, x_2, \cdots\}$, an infinite set. However, $S \cap X_i$ is finite for each i, hence closed in X_i. Since X has the weak topology, S is closed in X. By the same reasoning, every subset of S is closed, so S has the discrete topology. Since S is a closed subspace of the compact space C, it is compact. Hence S must be finite, a contradiction. $\square$

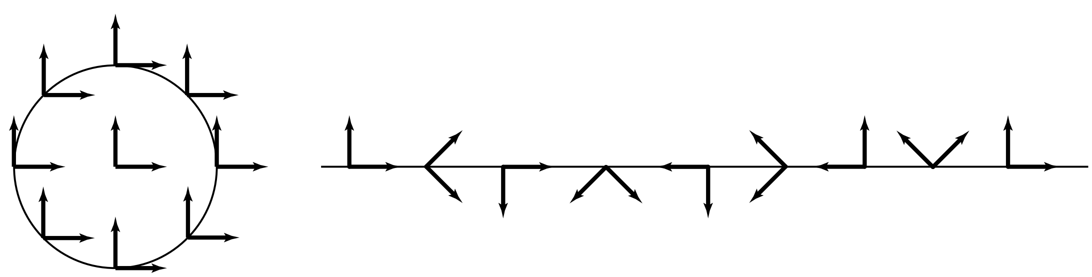
The constructions and results in this subsection hold equally well for vector bundles over $\mathbb{C}$, with $G_n(\mathbb{C}^k)$ the space of n-dimensional $\mathbb{C}$-linear subspaces of $\mathbb{C}^k$, etc. In particular, the proof of Theorem 1.7 translates directly to complex vector bundles, showing that $Vect^n_{\mathbb{C}}(X) \approx [X, G_n(\mathbb{C}^\infty)]$.

Vector Bundles over Spheres

Vector bundles with base space a sphere can be described more explicitly, and this will allow us to compute $Vect^n(S^k)$ for small values of k at least.

First let us describe a way to construct vector bundles $E \to S^k$. Write S^k as the union of its upper and lower hemispheres D^k_+ and D^k_-, with $D^k_+ \cap D^k_- = S^{k-1}$. Given a map $f : S^{k-1} \to GL_n(\mathbb{R})$, let E_f be the quotient of the disjoint union $D^k_+ \times \mathbb{R}^n \amalg D^k_- \times \mathbb{R}^n$ obtained by identifying $(x, v) \in \partial D^k_+ \times \mathbb{R}^n$ with $(x, f(x)(v)) \in \partial D^k_- \times \mathbb{R}^n$. The natural projection $E_f \to S^k$ is then a vector bundle. The map f is called its *clutching function*. (Presumably the terminology comes from the clutch which engages and disengages gears in machinery.)

Example 1. Let us see how the tangent bundle TS^2 to S^2 can be described in these terms. Define orthogonal vector fields v_+ and w_+ on the northern hemisphere D^2_+ by starting with a standard pair of orthogonal vectors at each point of a flat disk D^2 as in the left-hand figure below, then bending the disk over the northern hemisphere of S^2, carrying the vectors along as tangent vectors to the resulting curved disk. As we circumnavigate the equator of S^2, v_+ and w_+ then rotate through an angle of 2π relative to the equatorial direction, as in the right half of the figure.



Reflecting everything across the equatorial plane, we obtain orthogonal vector fields v_- and w_- on the southern hemisphere D^2_-. The restrictions of v_- and w_- to the equator also rotate through an angle of 2π, but in the opposite direction from v_+ and w_+ since we have reflected across the equator. Each pair $(v_\pm, w_\pm)$ defines a trivialization of TS^2 over $D^2_\pm$ taking $(v_\pm, w_\pm)$ to the standard basis for $\mathbb{R}^2$. Over the equator S^1 we then have two trivializations, and the function $f : S^1 \to GL_2(\mathbb{R})$ which rotates (v_+, w_+) to (v_-, w_-) sends $\theta \in S^1$, thought of as an angle, to rotation through the angle 2θ. For this map f we then have $E_f = TS^2$.

Example 2. The same definition of E_f works for complex vector bundles, starting with

$f : S^{k-1} \to GL_n(\mathbb{C})$, so let us interpret the canonical line bundle over $\mathbb{C}\mathrm{P}^1 = S^2$ in this light. (This example will play a crucial role in the next chapter.) The space $\mathbb{C}\mathrm{P}^1$ is the quotient of $\mathbb{C}^2 - \{0\}$ under the equivalence relation $(z_0, z_1) \sim \lambda(z_0, z_1)$. Denote the equivalence class of (z_0, z_1) by $[z_0, z_1]$. We can also write points of $\mathbb{C}\mathrm{P}^1$ as ratios $z = z_1/z_0 \in \mathbb{C} \cup \{\infty\} = S^2$. Points in the disk D^2_- inside the unit circle $S^1 \subset \mathbb{C}$ can be expressed uniquely in the form $[1, z_1/z_0] = [1, z]$ with $|z| \le 1$, and points in the disk D^2_+ outside S^1 can be written uniquely in the form $[z_0/z_1, 1] = [z^{-1}, 1]$ with $|z^{-1}| \le 1$. Over D^2_- a section of the canonical line bundle is then given by $[1, z_1/z_0] \mapsto (1, z_1/z_0)$ and over D^2_+ a section is $[z_0/z_1, 1] \mapsto (z_0/z_1, 1)$. These sections determine trivializations of the canonical line bundle over these two disks, and over their common boundary S^1 we pass from the D^2_+ trivialization to the D^2_- trivialization by multiplying by $z = z_1/z_0$. Thus the canonical line bundle is E_f for the clutching function $f : S^1 \to GL_1(\mathbb{C})$ defined by $f(z) = (z)$.

We return now to the general construction of bundles $E_f \to S^k$. A basic property of the clutching construction is that $E_f \approx E_g$ if $f \simeq g$. For if $F : S^{k-1} \times I \to GL_n(\mathbb{R})$ is a homotopy from f to g, then we can construct by the same method a vector bundle $E_F \to S^k \times I$ restricting to E_f over $S^k \times \{0\}$ and E_g over $S^k \times \{1\}$. Hence E_f and E_g are isomorphic by $(*)$ in the proof of Theorem 1.2.

Thus the association $[f] \mapsto E_f$ determines a well-defined map $\Phi : \pi_{k-1}(GL_n(\mathbb{R})) \longrightarrow Vect^n(S^k)$. If we change coordinates in $\mathbb{R}^n$ via a fixed $\alpha \in GL_n(\mathbb{R})$ we obtain an isomorphic bundle $E_{\alpha^{-1} f \alpha}$. Thus Φ induces a well-defined map on the set of orbits in $\pi_{k-1}(GL_n(\mathbb{R}))$ under the conjugation action of $GL_n(\mathbb{R})$, or what amounts to the same thing, the conjugation action of $\pi_0(GL_n(\mathbb{R}))$. Since $\pi_0(GL_n(\mathbb{R})) \approx \mathbb{Z}_2$ as we shall see below, we may write this set of orbits as $\pi_{k-1}(GL_n(\mathbb{R}))/\mathbb{Z}_2$.

Proposition 1.10. *The map* $\Phi : \pi_{k-1}(GL_n(\mathbb{R}))/\mathbb{Z}_2 \to Vect^n(S^k)$ *is a bijection.*

Proof: An inverse mapping Ψ can be constructed as follows. Given an n-dimensional vector bundle $p : E \to S^k$, its restrictions E_+ and E_- over D^k_+ and D^k_- are trivial since D^k_+ and D^k_- are contractible. Choose trivializations $h_\pm : E_\pm \to D^k_\pm \times \mathbb{R}^n$. Selecting a basepoint $s_0 \in S^{k-1}$ and fixing an isomorphism $p^{-1}(s_0) \approx \mathbb{R}^n$, we may assume h_+ and h_- are normalized to agree with this isomorphism on $p^{-1}(s_0)$. Then $h_- h_+^{-1}$ defines a map $(S^{k-1}, s_0) \to (GL_n(\mathbb{R}), \mathbb{1})$, whose homotopy class is by definition $\Psi(E) \in \pi_{k-1}(GL_n(\mathbb{R}))$. To see that $\Psi(E)$ is well-defined in the orbit set $\pi_{k-1}(GL_n(\mathbb{R}))/\mathbb{Z}_2$, note first that any two choices of normalized $h_\pm$ differ by a map $(D^k_\pm, s_0) \to (GL_n(\mathbb{R}), \mathbb{1})$. Since $D^k_\pm$ is contractible, such a map is homotopic to the constant map, so the two choices of $h_\pm$ are homotopic, staying fixed over s_0. Rechoosing the identification $p^{-1}(s_0) \approx \mathbb{R}^n$ has the effect of conjugating $\Psi(E)$ by an element of $GL_n(\mathbb{R})$, so $\Psi: Vect^n(S^k) \to \pi_{k-1}(GL_n(\mathbb{R}))/\mathbb{Z}_2$ is well-defined.

It is clear that Ψ and Φ are inverses of each other. $\square$

The case of complex vector bundles is similar but slightly simpler since $\pi_0(GL_n(\mathbb{C})) = 0$, and so we obtain bijections $Vect^n_{\mathbb{C}}(S^k) \approx \pi_{k-1}(GL_n(\mathbb{C}))$.

The same proof shows more generally that for a suspension SX with X paracompact, $Vect^n(SX) \approx \langle X, GL_n(\mathbb{R})\rangle/\mathbb{Z}_2$, where $\langle X, GL_n(\mathbb{R})\rangle$ denotes the basepoint-preserving homotopy classes of maps $X \to GL_n(\mathbb{R})$. Similarly $Vect^n_{\mathbb{C}}(SX) \approx \langle X, GL_n(\mathbb{C})\rangle$.

It is possible to compute a few homotopy groups of $GL_n(\mathbb{R})$ and $GL_n(\mathbb{C})$ by elementary means. The first observation is that $GL_n(\mathbb{R})$ deformation retracts onto the subgroup $O(n)$ consisting of orthogonal matrices, i.e., matrices whose columns form an orthonormal basis for $\mathbb{R}^n$, or in other words the matrices of isometries of $\mathbb{R}^n$ which fix the origin. The Gram-Schmidt process for converting a basis into an orthonormal basis provides a retraction of $GL_n(\mathbb{R})$ onto $O(n)$, continuity being evident from the explicit formulas for the Gram-Schmidt process. Each step of the process is in fact realizable by a homotopy, by inserting appropriate scalar factors into the formulas, and this yields a deformation retraction. (Alternatively, one can use the so-called polar decomposition of matrices to show that $GL_n(\mathbb{R})$ is in fact homeomorphic to the product of $O(n)$ with a Euclidean space.) The same reasoning shows that $GL_n(\mathbb{C})$ deformation retracts onto the unitary subgroup $U(n)$, consisting of matrices whose columns form an orthonormal basis for $\mathbb{C}^n$ with respect to the standard hermitian inner product. These are the isometries in $GL_n(\mathbb{C})$.

Next, there are fiber bundles

$$O(n-1) \longrightarrow O(n) \xrightarrow{p} S^{n-1} \qquad\qquad U(n-1) \longrightarrow U(n) \xrightarrow{p} S^{2n-1}$$

where p is the map obtained by evaluating an isometry at a chosen unit vector, for example $(1, 0, \cdots, 0)$. Local triviality for the first bundle can be shown as follows. We can view $O(n)$ as the Stiefel manifold $V_n(\mathbb{R}^n)$ by regarding the columns of an orthogonal matrix as an orthonormal n-frame. In these terms, the map p projects an n-frame onto its first vector. Given a vector $v_1 \in S^{n-1}$, extend this to an orthonormal n-frame $v_1, \cdots, v_n$. For unit vectors v near v_1, applying Gram-Schmidt to $v, v_2, \cdots, v_n$ produces a continuous family of orthonormal n-frames with first vector v. The last $n-1$ vectors of these frames give an orthonormal basis for $v^\perp$ varying continuously with v. Each such basis gives an identification of $v^\perp$ with $\mathbb{R}^{n-1}$, hence $p^{-1}(v)$ is identified with $V_{n-1}(\mathbb{R}^{n-1}) = O(n-1)$, giving the desired local trivialization. The same argument works in the unitary case.

From the long exact sequences of homotopy groups for these bundles we deduce immediately:

Proposition 1.11. *The map $\pi_i(O(n)) \to \pi_i(O(n+1))$ induced by the inclusion $O(n) \hookrightarrow O(n+1)$ is an isomorphism for $i < n-1$ and a surjection for $i = 2n-1$. Similarly, the inclusion $U(n) \hookrightarrow U(n+1)$ induces an isomorphism on π_i for $i < 2n$ and a surjection for $i = 2n$.* $\square$

Here are tables of some low-dimensional calculations:

$\pi_i O(n)$

		$i \longrightarrow$ 1	2	3	4	
n	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
$\downarrow$	1	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
	2	0	0	0	0	$\cdots$
	3	0	0	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}$	

$\pi_i U(n)$

		$i \longrightarrow$ 1	2	3	4	
n	0	0	0	0	0	$\cdots$
$\downarrow$	1	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\cdots$
	2	0	0	0	0	$\cdots$
	3	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\cdots$

Proposition 1.11 says that along each row in the first table, the groups stabilize once we pass the diagonal term $\pi_n O(n+1)$, and in the second table the rows stabilize even sooner. The stable groups are given by Bott Periodicity:

i mod 8	0	1	2	3	4	5	6	7
$\pi_i O(n)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\pi_i U(n)$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

The calculation in the first two tables can be obtained using the following homeomorphisms, together with the fact that the universal cover of $\mathbb{R}\mathrm{P}^3$ is S^3:

$$O(n) \approx S^0 \times SO(n) \qquad U(n) \approx S^1 \times SU(n)$$
$$SO(1) = \{1\} \qquad SU(1) = \{1\}$$
$$SO(2) \approx S^1 \qquad SU(2) \approx S^3$$
$$SO(3) \approx \mathbb{R}\mathrm{P}^3$$
$$SO(4) \approx \mathbb{R}\mathrm{P}^3 \times S^3$$

Here $SO(n)$ and $SU(n)$ are the subgroups consisting of matrices of determinant 1. A homeomorphism $O(n) \to S^0 \times SO(n)$ can be defined by $\alpha \mapsto (det(\alpha), \alpha')$ where α' is obtained from α by multiplying its last column by the scalar $1/det(\alpha)$. The inverse homeomorphism sends $(\lambda, \alpha) \in S^0 \times SO(n)$ to the matrix obtained by multiplying the last column of α by λ. The same formulas in the complex case give a homeomorphism $U(n) \approx S^1 \times SU(n)$.

It is obvious that $SO(1)$ and $SU(1)$ are trivial. For the homeomorphisms $SO(2) \approx S^1$ and $SU(2) \approx S^3$, note that 2×2 orthogonal or unitary matrices are determined by their first column, which can be any unit vector in $\mathbb{R}^2$ or $\mathbb{C}^2$.

A homeomorphism $SO(3) \approx \mathbb{R}\mathrm{P}^3$ can be obtained in the following way. Let $\varphi : D^3 \to SO(3)$ send a nonzero vector $x \in D^3$ to the rotation through angle $|x|\pi$ about the line determined by x. An orientation convention, such as the 'right-hand rule,' is needed to make this unambiguous. By continuity, φ must send 0 to the identity. Antipodal points of $S^2 = \partial D^3$ are sent to the same rotation through angle π, so φ induces a map $\overline{\varphi} : \mathbb{R}\mathrm{P}^3 \to SO(3)$, where $\mathbb{R}\mathrm{P}^3$ is viewed as D^3 with antipodal boundary points identified. The map $\overline{\varphi}$ is clearly injective since the axis of a nontrivial rotation is uniquely determined as its fixed point set, and $\overline{\varphi}$ is surjective since by easy linear algebra, each non-identity element of $SO(3)$ is a rotation about a unique axis. It follows that $\overline{\varphi}$ is a homeomorphism $\mathbb{R}\mathrm{P}^3 \approx SO(3)$.

It remains to show that $SO(4)$ is homeomorphic to $S^3 \times SO(3)$. Identifying $\mathbb{R}^4$ with the quaternions $\mathbb{H}$ and S^3 with the group of unit quaternions, the quaternion multiplication $w \mapsto vw$ for fixed $v \in S^3$ defines an isometry $\rho_v \in O(4)$ since quaternionic multiplication satisfies $|vw| = |v||w|$ and we are taking v to be a unit vector. Points of $O(4)$ can be viewed as 4-tuples $(v_1, \cdots, v_4)$ of orthonormal vectors $v_i \in \mathbb{H} = \mathbb{R}^4$, and $O(3)$ can be viewed as the subspace with $v_1 = 1$. Define a map $S^3 \times O(3) \to O(4)$ by sending $(v, (1, v_2, v_3, v_4))$ to (v, vv_2, vv_3, vv_4), the result of applying ρ_v to the orthonormal frame $(1, v_2, v_3, v_4)$. This map is a homeomorphism since it has an inverse defined by $(v, v_2, v_3, v_4) \mapsto (v, (1, v^{-1}v_2, v^{-1}v_3, v^{-1}v_4))$, the second coordinate being the orthonormal frame obtained by applying $\rho_{v^{-1}}$ to the frame (v, v_2, v_3, v_4). Since the path-components of $S^3 \times O(3)$ and $O(4)$ are homeomorphic to $S^3 \times SO(3)$ and $SO(4)$, respectively, we obtain a homeomorphism $S^3 \times SO(3) \approx SO(4)$.

The conjugation action of $\pi_0 O(n) \approx \mathbb{Z}_2$ on $\pi_i(O(n))$ which appears in the bijection $Vect^n(S^{i+1}) \approx \pi_i O(n)/\mathbb{Z}_2$ is trivial in the stable range $i < n - 1$ since we can realize each element of $\pi_i(O(n))$ by a map $S^i \to O(i+1)$ and then act on this by conjugating by a reflection across a hyperplane containing $\mathbb{R}^{i+1}$. Note that the map $Vect^n(S^{i+1}) \to Vect^{n+1}(S^{i+1})$ corresponding to the map $\pi_i O(n) \to \pi_i O(n+1)$ induced by the inclusion $O(n) \hookrightarrow O(n+1)$ is just direct sum with the trivial line bundle. Thus the stable isomorphism classes of vector bundles over spheres form groups, the same groups appearing in Bott Periodicity. This is the beginning of K-theory, as we shall see in the next chapter.

Outside the stable range the conjugation action is not always trivial. For example, in $\pi_1 O(2) \approx \mathbb{Z}$ the action is given by the nontrivial automorphism of $\mathbb{Z}$, multiplication by -1, since conjugating a rotation of $\mathbb{R}^2$ by a reflection produces a rotation in the opposite direction. Thus 2-dimensional vector bundles over S^2 are classified by nonnegative integers. When we stabilize by taking direct sum with a line bundle, then we are in the stable range where $\pi_1 O(n) \approx \mathbb{Z}_2$, so the 2-dimensional bundles corresponding to even integers are the ones which are stably trivial. The tangent bundle $T(S^2)$ is stably trivial, hence corresponds to an even integer, in fact to 2 as we saw in an example earlier in this section.

The two ways of classifying n-dimensional vector bundles over S^k, as $[S^k, G_n(\mathbb{R}^\infty)]$ and as $\pi_{k-1}O(n)/\mathbb{Z}_2$, can be related to each other in the following way. First, there is a fiber bundle $O(n) \to V_n(\mathbb{R}^\infty) \to G_n(\mathbb{R}^\infty)$ where the map $V_n \to G_n$ projects an n-frame onto the n-plane it spans. Local triviality follows from local triviality of the universal bundle $E_n \to G_n$ since V_n can be viewed as the bundle of n-frames in fibers of E_n. The space $V_n(\mathbb{R}^\infty)$ is contractible, as shown in an example in § 4.3 of [AT I]. At the moment we just need the fact that the homotopy groups of $V_n(\mathbb{R}^\infty)$ are trivial, and this follows from Proposition 1.11 since there are fiber bundles $O(k-n) \to O(k) \to V_n(\mathbb{R}^k)$. Since the homotopy groups of the total space of the fiber bundle $O(n) \to V_n(\mathbb{R}^\infty) \to G_n(\mathbb{R}^\infty)$ are trivial, we get isomorphisms $\pi_k G_n(\mathbb{R}^\infty) \approx \pi_{k-1}O(n)$. By Proposition 4A.1 of [AT I], $[S^k, G_n(\mathbb{R}^\infty)]$ is $\pi_k G_n(\mathbb{R}^\infty)$ modulo the action of $\pi_1 G_n(\mathbb{R}^\infty)$. Thus $Vect^n(S^k)$ is equal to both $\pi_k G_n(\mathbb{R}^\infty)$ modulo the action of $\pi_1 G_n(\mathbb{R}^\infty)$ and $\pi_{k-1}O(n)$ modulo the action of $\pi_0 O(n)$. One can check that under the isomorphisms $\pi_k G_n(\mathbb{R}^\infty) \approx \pi_{k-1}O(n)$ and $\pi_0 O(n) \approx \pi_1 G_n(\mathbb{R}^\infty)$ the actions correspond, so the two descriptions of $Vect^n(S^k)$ are equivalent.

Orientable Vector Bundles

An orientation of $\mathbb{R}^n$ is an equivalence class of ordered bases, two ordered bases being equivalent if the linear isomorphism taking one to the other has positive determinant. An *orientation* of an n-dimensional vector bundle is a choice of orientation in each fiber which is locally constant, in the sense that it is defined in a neighborhood of any fiber by n independent local sections.

Let $Vect_+^n(B)$ be the set of orientation-preserving isomorphism classes of oriented n-dimensional vector bundles over B. The proof of Theorem 1.7 extends without difficulty to show that $Vect_+^n(B) \approx [B, \widetilde{G}_n]$ where $\widetilde{G}_n$ is the space of oriented n-planes in $\mathbb{R}^\infty$, a 2-sheeted covering space of G_n, and the universal oriented bundle $\widetilde{E}_n$ over $\widetilde{G}_n$ consists of pairs $(\ell, v) \in \widetilde{G}_n \times \mathbb{R}^\infty$ with $v \in \ell$. In other words, $\widetilde{E}_n \to \widetilde{G}_n$ is the pullback of

$E_n \to G_n$ via the projection $\widetilde{G}_n \to G_n$. Clearly, an n-dimensional vector bundle $E \to B$ is orientable iff its classifying map $f: B \to G_n$ with $f^*(E_n) \approx E$ lifts to a map $\widetilde{f}: B \to \widetilde{G}_n$. In fact, each lift $\widetilde{f}$ corresponds to an orientation of E. The space $\widetilde{G}_n$ is path-connected, since G_n is connected and two points of $\widetilde{G}_n$ having the same image in G_n are oppositely oriented n-planes which can be joined by a path in $\widetilde{G}_n$ rotating the n-plane 180 degrees in an ambient $(n+1)$-plane, reversing its orientation. Since $\pi_1(G_n) \approx \pi_0 O(n) \approx \mathbb{Z}_2$, this implies that $\widetilde{G}_n$ is the universal cover of G_n.

The oriented version of Proposition 1.9 is a bijection $\pi_{k-1}SO(n) \approx Vect_+^n(S^k)$, proved in the same way. Since $\pi_0 SO(n) = 0$, there is no action to factor out.

Complex vector bundles are always orientable, when regarded as real vector bundles by restricting the scalar multiplication to $\mathbb{R}$. For if $v_1, \cdots, v_n$ is a basis for $\mathbb{C}^n$ then the basis $v_1, iv_1, \cdots, v_n, iv_n$ for $\mathbb{C}^n$ as an $\mathbb{R}$-vector space determines an orientation of $\mathbb{C}^n$ which is independent of the choice of $\mathbb{C}$-basis $v_1, \cdots, v_n$ since any other $\mathbb{C}$-basis can be joined to this one by a continuous path of $\mathbb{C}$-bases, the group $GL_n(\mathbb{C})$ being path-connected.

Cell Structure on Grassmannians

Since Grassmann manifolds play such a fundamental role in vector bundle theory, it would be good to have a better grasp on their topology. Here we show that $G_n(\mathbb{R}^\infty)$ has the structure of a CW complex with each $G_n(\mathbb{R}^k)$ a finite subcomplex. We will also see that $G_n(\mathbb{R}^k)$ is a closed manifold of dimension $n(k-n)$. Similar statements hold in the complex case as well.

For a start let us show that $G_n(\mathbb{R}^k)$ is Hausdorff, since we will need this fact later when we construct the CW structure. Given two n-planes ℓ and ℓ' in $G_n(\mathbb{R}^k)$, it suffices to find a continuous $f: G_n(\mathbb{R}^k) \to \mathbb{R}$ taking different values on ℓ and ℓ'. For a vector $v \in \mathbb{R}^k$ let $f_v(\ell)$ be the length of the orthogonal projection of v onto ℓ. This is a continuous function of ℓ since if we choose an orthonormal basis $v_1, \cdots, v_n$ for ℓ then $f_v(\ell) = \big((v \cdot v_1)^2 + \cdots + (v \cdot v_n)^2\big)^{1/2}$, which is certainly continuous in $v_1, \cdots, v_n$ hence in ℓ since $G_n(\mathbb{R}^k)$ has the quotient topology from $V_n(\mathbb{R}^k)$. If we choose $v \in \ell$ then $f_v(\ell) = |v|$, which is greater than $f_v(\ell')$ if $\ell' \neq \ell$.

In order to construct the CW structure we need some notation and terminology. In $\mathbb{R}^\infty$ we have the standard subspaces $\mathbb{R}^1 \subset \mathbb{R}^2 \subset \cdots$. For an n-plane $\ell \in G_n$ there is then the increasing chain of subspaces $\ell_j = \ell \cap \mathbb{R}^j$, with $\ell_j = \ell$ for large j. Each ℓ_j either equals ℓ_{j-1} or has dimension one greater than ℓ_{j-1} since ℓ_j is spanned by ℓ_{j-1} together with any vector in $\ell_j - \ell_{j-1}$. Let $\sigma_i(\ell)$ be the minimum j such that ℓ_j has dimension i. The increasing sequence $\sigma(\ell) = (\sigma_1(\ell), \cdots, \sigma_n(\ell))$ is called the *Schubert symbol* of ℓ. For example, if ℓ is the standard $\mathbb{R}^n \subset \mathbb{R}^\infty$ then $\ell_j = \mathbb{R}^j$ for $j \le n$ and $\sigma(\mathbb{R}^n) = (1, 2, \cdots, n)$. Clearly, $\mathbb{R}^n$ is the only n-plane with this Schubert symbol.

For a Schubert symbol $\sigma = (\sigma_1, \cdots, \sigma_n)$ let $e(\sigma) = \{\, \ell \in G_n \mid \sigma(\ell) = \sigma \,\}$.

Proposition 1.12. *$e(\sigma)$ is an open cell of dimension $(\sigma_1 - 1) + (\sigma_2 - 2) + \cdots + (\sigma_n - n)$, and these cells $e(\sigma)$ are the cells of a CW structure on G_n. The subspace $G_n(\mathbb{R}^k)$ is the finite subcomplex consisting of cells with $\sigma_n \le k$.*

For example $G_2(\mathbb{R}^4)$ has six cells corresponding to the Schubert symbols $(1,2)$, $(1,3)$, $(1,4)$, $(2,3)$, $(2,4)$, $(3,4)$, with the dimensions of these cells being $0, 1, 2, 2, 3, 4$ respectively.

Proof: Our main task will be to find a characteristic map for $e(\sigma)$. Note first that $e(\sigma) \subset G_n(\mathbb{R}^k)$ for $k \ge \sigma_n$. Let H_i be the hemisphere in $S^{\sigma_i - 1} \subset \mathbb{R}^{\sigma_i} \subset \mathbb{R}^k$ consisting of unit vectors with non-negative σ_i-th coordinate. In the Stiefel manifold $V_n(\mathbb{R}^k)$ let $E(\sigma)$ be the subspace of orthonormal frames $(v_1, \cdots, v_n) \in (S^{k-1})^n$ such that that $v_i \in H_i$ for each i. We claim that the projection $\pi : E(\sigma) \to H_1$, $\pi(v_1, \cdots, v_n) = v_1$, is a trivial fiber bundle. This is equivalent to finding a projection $p : E(\sigma) \to \pi^{-1}(v_0)$ which is a homeomorphism on fibers of π, where $v_0 = (0, \cdots, 0, 1) \in \mathbb{R}^{\sigma_1} \subset \mathbb{R}^k$, since the map $\pi \times p : E(\sigma) \to H_1 \times \pi^{-1}(v_0)$ is then a continuous bijection of compact Hausdorff spaces, hence a homeomorphism. The map $p : \pi^{-1}(v) \to \pi^{-1}(v_0)$ is obtained by applying the rotation ρ_v of $\mathbb{R}^k$ taking v to v_0 and fixing the $(k-2)$-dimensional subspace orthogonal to v and v_0. This rotation takes H_i to itself for $i > 1$ since it affects only the first σ_1 coordinates of vectors in $\mathbb{R}^k$. Hence p takes $\pi^{-1}(v)$ onto $\pi^{-1}(v_0)$.

The fiber $\pi^{-1}(v_0)$ can be identified with $E(\sigma')$ for $\sigma' = (\sigma_2 - 1, \cdots, \sigma_n - 1)$. By induction on n this is homeomorphic to a closed ball of dimension $(\sigma_2 - 2) + \cdots + (\sigma_n - n)$, so $E(\sigma)$ is a closed ball of dimension $(\sigma_1 - 1) + \cdots + (\sigma_n - n)$.

The natural map $E(\sigma) \to G_n$ taking an orthonormal n-tuple to the n-plane it spans maps the interior of the ball $E(\sigma)$ to $e(\sigma)$ bijectively, since each $\ell \in e(\sigma)$ has a unique basis $(v_1, \cdots, v_n) \in int(E(\sigma))$. Namely, consider the sequence of subspaces $\ell_{\sigma_1} \subset \cdots \subset \ell_{\sigma_n}$, and choose $v_i \in \ell_{\sigma_i}$ to be the unit vector with positive σ_i-th coordinate orthogonal to $\ell_{\sigma_{i-1}}$. Since G_n has the quotient topology from V_n, the map $int(E(\sigma)) \to e(\sigma)$ is a homeomorphism, so $e(\sigma)$ is an open cell of dimension $(\sigma_1 - 1) + \cdots + (\sigma_n - n)$. The boundary of $E(\sigma)$ maps to cells $e(\sigma')$ of G_n where σ' is obtained from σ by decreasing some σ_i's, so these cells $e(\sigma')$ have lower dimension than $e(\sigma)$.

It is clear from the definitions that $G_n(\mathbb{R}^k)$ is the union of the cells $e(\sigma)$ with $\sigma_n \le k$. To see that the maps $E(\sigma) \to G_n(\mathbb{R}^k)$ for these cells are the characteristic maps for a CW structure on $G_n(\mathbb{R}^k)$ we can argue as follows. For fixed k, let X^i be the union of the cells $e(\sigma)$ in $G_n(\mathbb{R}^k)$ having dimension at most i. Suppose by induction on i that X^i is a CW complex with these cells. Attaching the $(i+1)$-cells $e(\sigma)$ of X^{i+1} to X^i via the maps $\partial E(\sigma) \to X^i$ produces a CW complex Y and a natural continuous surjection $Y \to X^{i+1}$. Since Y is a finite CW complex, it is compact and Hausdorff. The space X^{i+1} is the continuous image of Y so it is compact, and it is Hausdorff as a subspace of $G_n(\mathbb{R}^k)$. So the map $Y \to X^{i+1}$ is a homeomorphism, and X^{i+1} is a CW complex, finishing the induction. Thus we have a CW structure on $G_n(\mathbb{R}^k)$.

Since the inclusions $G_n(\mathbb{R}^k) \subset G_n(\mathbb{R}^{k+1})$ for varying k are inclusions of subcomplexes, and $G_n(\mathbb{R}^\infty)$ has the weak topology with respect to these subspaces, it follows that we have a CW structure on $G_n(\mathbb{R}^\infty)$. □

Similar constructions work to give CW structures on complex Grassmannians, but here $e(\sigma)$ will be a cell of dimension $(2\sigma_1 - 2) + (2\sigma_2 - 4) + \cdots + (2\sigma_n - 2n)$. The 'hemisphere' H_i is defined to be the subspace of the unit sphere $S^{2\sigma_i - 1}$ in $\mathbb{C}^{\sigma_i}$ consisting of vectors whose σ_i-th coordinate is non-negative real, so H_i is a ball of dimension $2\sigma_i - 2$. The transformation $\rho_v \in SU(k)$ is uniquely determined by specifying that it takes v to v_0 and fixes the orthogonal $(k-2)$-dimensional complex subspace, since an element of $U(2)$ of determinant 1 is determined by where it sends one unit vector.

The highest-dimensional cell of $G_n(\mathbb{R}^k)$ is $e(\sigma)$ for $\sigma = (k-n+1, k-n+2, \cdots, k)$, of dimension $n(k-n)$, so this is the dimension of $G_n(\mathbb{R}^k)$. Near points in these top-dimensional cells $G_n(\mathbb{R}^k)$ is a manifold. But $G_n(\mathbb{R}^k)$ is homogeneous in the sense that given any two points in $G_n(\mathbb{R}^k)$ there is a homeomorphism $G_n(\mathbb{R}^k) \to G_n(\mathbb{R}^k)$ taking one point to the other, namely, the homeomorphism induced by an invertible linear map $\mathbb{R}^k \to \mathbb{R}^k$ taking one n-plane to the other. From this homogeneity it follows that $G_n(\mathbb{R}^k)$ is a manifold near all points. Since it is compact, it is therefore a closed manifold.

There is a natural inclusion $i : G_n \hookrightarrow G_{n+1}$, $i(\ell) = \mathbb{R} \times j(\ell)$ where $j : \mathbb{R}^\infty \to \mathbb{R}^\infty$ is the embedding $j(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots)$. If $\sigma(\ell) = (\sigma_1, \cdots, \sigma_n)$, then $\sigma(i(\ell)) = (1, \sigma_1 + 1, \cdots, \sigma_n + 1)$, so i takes cells of G_n to cells of G_{n+1} of the same dimension, making $i(G_n)$ a subcomplex of G_{n+1}. Identifying G_n with the subcomplex $i(G_n)$, we obtain an increasing sequence of CW complexes $G_1 \subset G_2 \subset \cdots$ whose union $\bigcup_n G_n$ is therefore a CW complex which we may denote G_∞. Similar remarks apply as well in the complex case.

Appendix: Paracompactness

A Hausdorff space X is *paracompact* if every open cover $\{U_\alpha\}$ of X has a subordinate partition of unity $\{\varphi_\beta\}$, i.e., the φ_β's are maps $X \to I$, each φ_β has support (the closure of the set where $\varphi_\beta \neq 0$) contained in some U_α, each $x \in X$ has a neighborhood in which only finitely many φ_β's are nonzero, and $\sum_\beta \varphi_\beta = 1$. An equivalent definition which is often given is that X is Hausdorff and every open cover of X has a locally finite open refinement. The first definition clearly implies the second, by taking the cover $\{\varphi_\beta^{-1}(0,1]\}$. For the converse, see [Dugundji] or [Lundell-Weingram]. It is the former definition which is most useful in algebraic topology, and the fact that the two definitions are equivalent is rarely if ever needed. So we shall use the first definition.

A paracompact space X is normal, for let A_1 and A_2 be disjoint closed sets in X, and let $\{\varphi_\beta\}$ be a partition of unity subordinate to the cover $\{X - A_1, X - A_2\}$. Let φ_i be the sum of the φ_β's which are nonzero at some point of A_i. Then $\varphi_i(A_i) = 1$, and $\varphi_1 + \varphi_2 \leq 1$ since no φ_β can be a summand of both φ_1 and φ_2. Hence $\varphi_1^{-1}(1/2, 1]$ and $\varphi_2^{-1}(1/2, 1]$ are disjoint open sets containing A_1 and A_2, respectively.

Most of the spaces one meets in algebraic topology are paracompact, including:

(1) compact Hausdorff spaces.

(2) unions of increasing sequences $X_1 \subset X_2 \subset \cdots$ of compact Hausdorff spaces X_i, with the weak topology (as in the definition of CW complexes).

(3) CW complexes.

(4) metric spaces.

Note that (2) includes (3) for CW complexes with countably many cells, since such a CW complex can be expressed as an increasing union of finite subcomplexes. Using (1) and (2), it can be shown that many manifolds are paracompact, for example $\mathbb{R}^n$ itself.

The next three propositions verify that the spaces in (1), (2), and (3) are paracompact.

Proposition 1.13. *A compact Hausdorff space X is paracompact.*

Proof: Let $\{U_\alpha\}$ be an open cover of X. For each $x \in U_\alpha$ let V_x be an open neighborhood of x with closure contained in U_α. By the Tietze extension theorem, or Uryssohn's lemma, there is a map $\varphi_x : X \to I$ with $\varphi_x(x) = 1$ and $\varphi_x(X - V_x) = 0$. The open cover $\{\varphi_x^{-1}(0,1]\}$ of X contains a finite subcover, and we relabel the corresponding φ_x's as φ_β. Then $\sum_\beta \varphi_\beta(x) > 0$ for all x, and we obtain the desired partition of unity subordinate to $\{U_\alpha\}$ by normalizing each φ_β by dividing it by $\sum_\beta \varphi_\beta$. □

Proposition 1.14. *If X is the direct limit of an increasing sequence $X_1 \subset X_2 \subset \cdots$ of compact Hausdorff spaces X_i, then X is paracompact.*

Proof: A preliminary observation is that X is normal. To show this, it suffices to find a map $f\colon X \to I$ with $f(A)=0$ and $f(B)=1$ for any two disjoint closed sets A and B. Such an f can be constructed inductively over the X_i's, using normality of the X_i's. For the induction step one has f defined on the closed set $X_i \cup (A\cap X_{i+1}) \cup (B \cap X_{i+1})$ and one extends over X_{i+1} by Tietze's theorem.

To prove that X is paracompact, let an open cover $\{U_\alpha\}$ be given. Since X_i is compact Hausdorff, there is a finite partition of unity $\{\varphi_{ij}\}$ on X_i subordinate to $\{U_\alpha \cap X_i\}$. Using normality of X, extend each φ_{ij} to a map $\varphi_{ij}\colon X \to I$ with support in the same U_α. Let $\sigma_i = \sum_j \varphi_{ij}$. This sum is 1 on X_i, so if we normalize each φ_{ij} by dividing it by $max\{1/2, \sigma_i\}$, we get new maps φ_{ij} with $\sigma_i = 1$ in a neighborhood V_i of X_i. Let $\psi_{ij} = max\{0, \varphi_{ij} - \sum_{k<i} \sigma_k\}$. Since $0 \le \psi_{ij} \le \varphi_{ij}$, the collection $\{\psi_{ij}\}$ is subordinate to $\{U_\alpha\}$. In V_i all ψ_{kj}'s with $k > i$ are zero, so each point of X has a neighborhood in which only finitely many ψ_{ij}'s are nonzero. For each $x \in X$ there is a ψ_{ij} with $\psi_{ij}(x) > 0$, since if $\varphi_{ij}(x) > 0$ and i is minimal with respect to this condition, then $\psi_{ij}(x) = \varphi_{ij}(x)$. Thus when we normalize the collection $\{\psi_{ij}\}$ by dividing by $\sum \psi_{ij}$ we obtain a partition of unity on X subordinate to $\{U_\alpha\}$. □

Proposition 1.15. *Every CW complex is paracompact.*

Proof: Given an open cover $\{U_\alpha\}$ of the CW complex X, suppose inductively that we have a partition of unity $\{\varphi_\beta\}$ on X^n subordinate to the cover $\{U_\alpha \cap X^n\}$. For a cell e^{n+1}_γ with characteristic map $\Phi_\gamma\colon D^{n+1} \to X$, $\{\varphi_\beta \Phi_\gamma\}$ is a partition of unity on $S^n = \partial D^{n+1}$. Since S^n is compact, only finitely many of these compositions $\varphi_\beta\Phi_\gamma$ can be nonzero, for fixed γ. We extend these functions $\varphi_\beta\Phi_\gamma$ over D^{n+1} by the formula $\rho_\varepsilon(r)\varphi_\beta\Phi_\gamma(x)$ where $(r,x) \in I \times S^n$ are 'spherical' coordinates in D^{n+1} and $\rho_\varepsilon\colon I \to I$ is 0 on $[0, 1-\varepsilon]$ and 1 on $[1-\varepsilon/2, 1]$. If $\varepsilon = \varepsilon_\gamma$ is chosen small enough, these extended functions $\rho_\varepsilon\varphi_\beta\Phi_\gamma$ will be subordinate to the cover $\{\Phi_\gamma^{-1}(U_\alpha)\}$. Let $\{\psi_{\gamma j}\}$ be a finite partition of unity on D^{n+1} subordinate to $\{\Phi_\gamma^{-1}(U_\alpha)\}$. Then $\{\rho_\varepsilon\varphi_\beta\Phi_\gamma, (1-\rho_\varepsilon)\psi_{\gamma j}\}$ is a partition of unity on D^{n+1} subordinate to $\{\Phi_\gamma^{-1}(U_\alpha)\}$, extending the partition of unity $\{\varphi_\beta\Phi_\gamma\}$ on S^n. Via Φ_γ^{-1} this gives an extension of $\{\varphi_\beta\}$ over each cell e^{n+1}_γ, hence over X^{n+1}. After we make such extensions for all n, we obtain a partition of unity $\{\varphi_\beta\}$ subordinate to $\{U_\alpha\}$, since by construction, each point of X has a neighborhood in which only finitely many of the final φ_β's are nonzero. □

Here is an occasionally useful technical fact about paracompact spaces:

Lemma 1.16. *Given an open cover $\{U_\alpha\}$ of the paracompact space X, there is a countable open cover $\{V_k\}$ such that each V_k is a disjoint union of open sets each contained in* *some U_α, and there is a partition of unity $\{\varphi_k\}$ with φ_k supported in V_k.*

Proof: Let $\{\varphi_\beta\}$ be a partition of unity subordinate to $\{U_\alpha\}$. For each finite set S of functions φ_β let V_S be the subset of X where all the φ_β's in S are strictly greater than all the φ_β's not in S. Since only finitely many φ_β's are nonzero near any $x \in X$, V_S is defined by finitely many inequalities among φ_β's near x, so V_S is open. Also, V_S is contained in some U_α, namely, any U_α containing the support of any $\varphi_\beta \in S$, since $\varphi_\beta \in S$ implies $\varphi_\beta > 0$ on V_S. Let V_k be the union of all the open sets V_S such that S has k elements. This is clearly a disjoint union. The collection $\{V_k\}$ is a cover of X since if $x \in X$ then $x \in V_S$ where $S = \{\, \varphi_\beta \mid \varphi_\beta(x) > 0 \,\}$.

For the second statement, let $\{\varphi_\gamma\}$ be a partition of unity subordinate to the cover $\{V_k\}$, and let φ_k be the sum of those φ_γ's supported in V_k but not in V_j for $j < k$. $\square$

Exercises

1. Show that a vector bundle $E \to X$ has k independent sections iff it has a trivial k-dimensional subbundle.

2. For a vector bundle $E \to X$ with a subbundle $E' \subset E$, construct a quotient vector bundle $E/E' \to X$.

3. Show that the orthogonal complement of a subbundle is independent of the choice of inner product, up to isomorphism.

4. A *vector bundle map* is a commutative diagram

$$\begin{array}{ccc} E' & \xrightarrow{\widetilde{f}} & E \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

where the two vertical maps are vector bundle projections and $\widetilde{f}$ is an isomorphism on each fiber. Given such a bundle map, show that E' is isomorphic to the pullback bundle $f^*(E)$.

5. For a vector bundle $E \to X$ and a closed subset $A \subset X$, show that a section $s : A \to E$ extends to a section $U \to E$ for some neighborhood U of A. Deduce that if E is trivial over A, it is trivial over some neighborhood of A. [Hint: Use the Tietze extension theorem .X is assumed paracompact, hence normal.]

6. For a closed contractible subspace $A \subset X$, show the quotient map $q : X \to X/A$ induces a bijection between the isomorphism classes of vector bundles over X and X/A. [Use the previous problem to show $E \mapsto q^*(E)$ is surjective.]

7. Show that the projection $V_n(R^k) \to G_n(\mathbb{R}^k)$ is a fiber bundle with fiber $O(n)$ by showing that it is the orthonormal n-frame bundle associated to the vector bundle $E_n(\mathbb{R}^k) \to G_n(\mathbb{R}^k)$.

8. Show that the pair $(G_n(\mathbb{R}^k), G_n(\mathbb{R}^\infty))$ is $(k-n)$-connected, and deduce that Proposition 1.8 holds for finite-dimensional CW complexes. [The lowest-dimensional cell of $G_n(\mathbb{R}^{k+1}) - G_n(\mathbb{R}^k)$ is the $e(\sigma)$ with $\sigma = (1, 2, \cdots, n-1, k+1)$, and this cell has dimension $k+1-n$.]

Chapter 2. Complex K-theory

The idea of K-theory is to make the direct sum operation on real or complex vector bundles over a fixed base space X into the addition operation in a group. There are two slightly different ways of doing this, producing, in the case of complex vector bundles, groups $K(X)$ and $\widetilde{K}(X)$ with $K(X) \approx \widetilde{K}(X) \oplus \mathbb{Z}$, and for real vector bundles, groups $KO(X)$ and $\widetilde{KO}(X)$ with $KO(X) \approx \widetilde{KO}(X) \oplus \mathbb{Z}$. Complex K-theory turns out to be somewhat simpler than real K-theory, so we concentrate on this case in the present chapter.

Computing $\widetilde{K}(X)$ even for simple spaces X requires some work. The case $X = S^n$ involves the Bott Periodicity Theorem, proved in §2.2. This is a deep theorem, so it is not surprising that it has applications of real substance, and we give some of these in §2.3, notably Adams' theorem on the Hopf invariant with its corollary on the nonexistence of division algebras over $\mathbb{R}$ in dimensions other than 1, 2, 4, and 8, the dimensions of the real and complex numbers, quaternions, and Cayley octonions. A further application to the J-homomorphism is delayed until the next chapter when we combine K-theory with ordinary cohomology.

1. The Functor K(X)

Since K-theory for complex vector bundles turns out to be simpler than for real vectors bundles, let us take "vector bundle" to mean generally "complex vector bundle" in this chapter. Base spaces will always be assumed paracompact, in particular Hausdorff, so that the results of Chapter 1 which presume paracompactness will be available to us.

Consider vector bundles over a fixed base space X. The trivial n-dimensional vector bundle we write as $\varepsilon^n \to X$. Define two equivalence relations on vector bundles over X by setting $E_1 \sim E_2$ if $E_1 \oplus \varepsilon^m \approx E_2 \oplus \varepsilon^n$ for some m and n, and $E_1 \approx_s E_2$ if $E_1 \oplus \varepsilon^n \approx E_2 \oplus \varepsilon^n$ for some n. On equivalence classes of either sort the operation of direct sum is well-defined, commutative, associative, and has a "zero," the class of ε^0.

Proposition 2.1. *If X is compact Hausdorff, the set of $\sim$-equivalence classes of vector bundles over X forms an abelian group with respect to $\oplus$.*

This group is called $\widetilde{K}(X)$.

Proof: Only the existence of inverses needs to be shown, which we do by showing that for each vector bundle $\pi : E \to X$ there is a bundle $E' \to X$ such that $E \oplus E' \approx \varepsilon^m$ for some m. If all the fibers of E have the same dimension, then Proposition 1.8 gives the result. To reduce to this case, let $X_i = \{\, x \in X \mid dim\, \pi^{-1}(x) = i \,\}$. These X_i's are disjoint open sets in X, hence are finite in number by compactness. By adding to E a bundle which over each X_i is a trivial bundle of suitable dimension we can produce a bundle whose fibers all have the same dimension. $\square$

For the direct sum operation on $\approx_s$-equivalence classes, only the "0" can have an inverse since $E \oplus E' \approx_s \varepsilon^0$ implies $E \oplus E' \oplus \varepsilon^n \approx \varepsilon^n$ for some n, which can only happen if E and E' are 0-dimensional. However, even though inverses do not exist, we do have the cancellation property that $E_1 \oplus E_2 \approx_s E_1 \oplus E_3$ implies $E_2 \approx_s E_3$ over a compact base space X, since we can add to both sides of $E_1 \oplus E_2 \approx_s E_1 \oplus E_3$ a bundle E_1' such that $E_1 \oplus E_1'$ is trivial.

Just as the positive rational numbers are constructed from the positive integers by forming quotients a/b with the equivalence relation $a/b = c/d$ iff $ad = bc$, so we can form for compact X an abelian group $K(X)$ consisting of formal differences $E - E'$ of vector bundles E and E' over X, with the equivalence relation $E_1 - E_1' = E_2 - E_2'$ if $E_1 \oplus E_2' \approx_s E_2 \oplus E_1'$. Verifying transitivity of this relation involves the cancellation property, which is why compactness of X is needed. With the obvious addition rule $E_1 - E_1' + E_2 - E_2' = (E_1 \oplus E_2) - (E_1' \oplus E_2')$, $K(X)$ is then a group. The zero element is the equivalence class of $E - E$ for any E, and the inverse of $E - E'$ is $E' - E$. Note that every element of $K(X)$ can be represented as a difference $E - \varepsilon^n$ since if we start with $E - E'$ we can add to both E and E' a bundle E'' such that $E' \oplus E''$ is trivial.

There is a natural homomorphism $K(X) \to \widetilde{K}(X)$ sending $E - \varepsilon^n$ to the $\sim$-class of E. This is well-defined since if $E - \varepsilon^n = E' - \varepsilon^m$ in $K(X)$, then $E \oplus \varepsilon^m \approx_s E' \oplus \varepsilon^n$, hence $E \sim E'$. This map $K(X) \to \widetilde{K}(X)$ is obviously surjective, and its kernel consists of elements $E - \varepsilon^n$ with $E \sim \varepsilon^0$, hence $E \approx_s \varepsilon^m$ for some m, so the kernel consists of the elements of the form $\varepsilon^m - \varepsilon^n$. This subgroup $\{\varepsilon^m - \varepsilon^n\}$ of $K(X)$ is isomorphic to $\mathbb{Z}$, and in fact restriction of vector bundles to a basepoint $x_0 \in X$ defines a homomorphism $K(X) \to K(x_0) \approx \mathbb{Z}$ which restricts to an isomorphism on the subgroup $\{\varepsilon^m - \varepsilon^n\}$. Thus we have a splitting $K(X) \approx \widetilde{K}(X) \oplus \mathbb{Z}$, depending on the choice of x_0. The group $\widetilde{K}(X)$ is sometimes called *reduced*, to distinguish it from $K(X)$.

Let us compute a few examples. The complex version of Proposition 1.9 gives a bijection between the set $Vect^k_{\mathbb{C}}(S^n)$ of isomorphism classes of k-dimensional vector bundles over S^n with $\pi_{n-1}U(k)$. Under this bijection, adding a trivial line bundle corresponds to including $U(k)$ in $U(k+1)$ by adjoining an $(n+1)^{st}$ row and column consisting of zeros except for a single 1 on the diagonal. Let $U = \bigcup_k U(k)$ with the weak topology: a subset of U is open iff it intersects each $U(k)$ in an open set in $U(k)$. This implies that each compact subset of U is contained in some $U(k)$, and it follows that the bijections $Vect^k_{\mathbb{C}}(S^n) \approx \pi_{n-1}U(k)$ induce a bijection $\widetilde{K}(S^n) \approx \pi_{n-1}U$.

Proposition 2.2. *This bijection is a group isomorphism* $\widetilde{K}(S^n) \approx \pi_{n-1}U$.

Proof: It remains to see that the two group operations correspond. Represent two elements of $\pi_{n-1}U$ by maps $f, g : S^{n-1} \to U(k)$ taking the basepoint of S^{n-1} to the identity matrix. The sum in $\widetilde{K}(S^n)$ then corresponds to the map $f \oplus g : S^{n-1} \to U(2k)$ having the matrices $f(x)$ in the upper left $k \times k$ block and the matrices $g(x)$ in the lower right $k \times k$ block, the other two blocks being zero. Since $\pi_0 U(2k) = 0$, there is a path $\alpha_t \in U(2k)$ from the identity to the matrix of the transformation which interchanges the two factors of $\mathbb{C}^k \times \mathbb{C}^k$. Then the matrix product $f \alpha_t g \alpha_t$ gives a homotopy from $f \oplus g$ to $fg \oplus \mathbb{1}$. It remains to see that the matrix product fg represents the sum $[f] + [g]$ in $\pi_{n-1}U(k)$.

This is a general fact about H-spaces which can be seen in the following way. The standard definition of the sum in $\pi_{n-1}U(k)$ is $[f] + [g] = [f + g]$ where the map $f + g$ consists of a compressed version of f on one hemisphere of S^{n-1} and a compressed version of g on the other. We can realize this map $f + g$ as a product $f_1 g_1$ of maps $S^{n-1} \to U(k)$ each mapping one hemisphere to the identity. There are homotopies f_t from $f = f_0$ to f_1 and g_t from $g = g_0$ to g_1. Then $f_t g_t$ is a homotopy from fg to $f_1 g_1 = f + g$. $\square$

This proposition generalizes easily to suspensions: For all compact X, $\widetilde{K}(SX)$ is isomorphic to $\langle X, U \rangle$, the group of basepoint-preserving homotopy classes of maps $X \to U$.

From the calculations of $\pi_i U$ in §1.2 we deduce that $\widetilde{K}(S^n)$ is 0, $\mathbb{Z}$, 0, $\mathbb{Z}$ for $n = 1$, 2, 3, 4. This alternation of 0's and $\mathbb{Z}$'s continues for all higher dimensional spheres:

Bott Periodicity Theorem. *There are isomorphisms* $\widetilde{K}(S^n) \approx \widetilde{K}(S^{n+2})$ *for all* $n \geq 0$. *More generally, there are isomorphisms* $\widetilde{K}(X) \approx \widetilde{K}(S^2 X)$ *for all compact* X, *where* $S^2 X$ *is the double suspension of* X.

The theorem actually says that a certain natural map $\beta : \widetilde{K}(X) \to \widetilde{K}(S^2 X)$ defined later in this section is an isomorphism. There is an equivalent form of Bott periodicity involving $K(X)$ rather than $\widetilde{K}(X)$, an isomorphism $\mu : K(X) \otimes K(S^2) \xrightarrow{\approx} K(X \times S^2)$. The map μ is easier to define than β, so this is what we will do next. Then we will set up some formal machinery which in particular shows that the two versions of Bott Periodicity are equivalent. The second version is the one which will be proved in §2.2.

Ring Structure

The groups $K(X)$ have a natural ring structure whose multiplication comes from tensor product of vector bundles. For elements of $K(X)$ represented by vector bundles E_1 and E_2 their product in $K(X)$ will be represented by the bundle $E_1 \otimes E_2$, so for arbitrary elements of $K(X)$ represented by differences of vector bundles, their product in $K(X)$ is defined by the formula:

$$(E_1 - E_1')(E_2 - E_2') = E_1 \otimes E_2 - E_1 \otimes E_2' - E_1' \otimes E_2 + E_1' \otimes E_2'.$$

It is routine to verify that this is well-defined and makes $K(X)$ into a commutative ring with identity ε^1, the trivial line bundle, using the basic properties of tensor product of vector bundles described in §1.1. We may as well simplify notation at this point by writing the element $\varepsilon^n \in K(X)$ simply as n. This is consistent with familiar arithmetic rules, for example, that the product nE is the sum of n copies of E.

If we choose a basepoint $x_0 \in X$, then the map $K(X) \to K(x_0)$ obtained by restricting vector bundles over x_0 is a ring homomorphism, so its kernel, which can be identified with $\widetilde{K}(X)$, is an ideal, hence also a ring in its own right, though not necessarily a ring with identity.

Example. Let us compute the ring structure in $K(S^2)$. As an abelian group, $K(S^2)$ is isomorphic to $\widetilde{K}(S^2) \oplus \mathbb{Z} \approx \mathbb{Z} \oplus \mathbb{Z}$, with additive basis $\{1, H\}$ where H is the canonical line bundle over $\mathbb{C}\mathrm{P}^1 = S^2$, by Proposition 2.2 and the calculations in §1.2. We use the notation "H" for the canonical line bundle over $\mathbb{C}\mathrm{P}^1$ since its unit sphere bundle is the Hopf bundle $S^3 \to S^2$. To determine the ring structure in $K(S^2)$ we have only to express the element H^2, represented by the tensor product $H \otimes H$, as a linear combination of 1 and H. The claim is that the bundle $(H \otimes H) \oplus 1$ is isomorphic to $H \oplus H$. This can be seen by looking at the clutching functions for these two bundles, which are the maps $S^1 \to U(2)$ given by

$$z \mapsto \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad z \mapsto \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$$

In the notation of the proof of Proposition 2.2, these are the clutching functions $fg \oplus \mathbb{1}$ and $f \oplus g$ where both f and g are the function $z \mapsto (z)$. As we showed, the clutching functions $fg \oplus \mathbb{1}$ and $f \oplus g$ are always homotopic, so this gives the desired isomorphism $(H \otimes H) \oplus 1 \approx H \oplus H$. In $K(S^2)$ this is the formula $H^2 + 1 = 2H$, so $H^2 = 2H - 1$. We can also write this as $(H-1)^2 = 0$, and then $K(S^2)$ can be described as the quotient $\mathbb{Z}[H]/(H-1)^2$ of the polynomial ring $\mathbb{Z}[H]$ by the ideal generated by $(H-1)^2$.

Note that if we regard $\widetilde{K}(S^2)$ as the kernel of $K(S^2) \to K(x_0)$, then it is generated as an abelian group by $H - 1$. Since we have the relation $(H-1)^2 = 0$, this means that the multiplication in $\widetilde{K}(S^2)$ is completely trivial: The product of any two elements is zero. Readers familiar with cup product in ordinary cohomology will recognize that the situation is exactly the same as in $H^*(S^2;\mathbb{Z})$ and $\widetilde{H}^*(S^2;\mathbb{Z})$, where $H-1$ behaves exactly like the generator of $H^2(S^2;\mathbb{Z})$. Of course, with ordinary cohomology the cup product of a generator of $H^2(S^2;\mathbb{Z})$ with itself is automatically zero since $H^4(S^2;\mathbb{Z})=0$, whereas with K-theory a calculation is required.

We consider next the functorial properties of $K(X)$ and $\widetilde{K}(X)$. A map $f:X\to Y$ induces a map $f^*:K(Y)\to K(X)$, sending $E-E'$ to $f^*(E)-f^*(E')$. This is a ring homomorphism since $f^*(E_1\oplus E_2)\approx f^*(E_1)\oplus f^*(E_2)$ and $f^*(E_1\otimes E_2)\approx f^*(E_1)\otimes f^*(E_2)$. The functor properties $(fg)^*=g^*f^*$ and $\mathbb{1}^*=\mathbb{1}$, as well as the fact that $f\simeq g$ implies $f^*=g^*$, follow from the corresponding properties for pullbacks of vector bundles. Similarly, we have induced maps $f^*:\widetilde{K}(Y)\to\widetilde{K}(X)$ with the same properties, except that for f^* to be a ring homomorphism we must be in the category of basepointed spaces and basepoint-preserving maps, since our definition of multiplication for $\widetilde{K}$ required basepoints.

An "external" product $\mu:K(X)\otimes K(Y)\to K(X\times Y)$ can be defined by $\mu(a\otimes b)=p_1^*(a)p_2^*(b)$ where p_1 and p_2 are the projections of $X\times Y$ onto X and Y. Taking Y to be S^2, we have the map $\mu:K(X)\otimes K(S^2)\to K(X\times S^2)$ which the alternative form of Bott Periodicity asserts to be an isomorphism.

Cohomological Properties

The reduced groups $\widetilde{K}$ satisfy a key exactness property:

Proposition 2.3. *If X is a finite cell complex and $A\subset X$ is a subcomplex, then the inclusion and quotient maps $A\xrightarrow{i}X\xrightarrow{q}X/A$ induce an exact sequence $\widetilde{K}(X/A)\xrightarrow{q^*}\widetilde{K}(X)\xrightarrow{i^*}\widetilde{K}(A)$.*

Proof: The inclusion $Im(q^*)\subset Ker(i^*)$ is equivalent to $i^*q^*=0$. Since qi is equal to the composition $A\to A/A\hookrightarrow X/A$ and $\widetilde{K}(A/A)=0$, it follows that $i^*q^*=0$.

For the opposite inclusion $Ker(i^*)\subset Im(q^*)$, suppose the restriction over A of a vector bundle $p:E\to X$ is stably trivial. Adding a trivial bundle to E, we may assume that E itself is trivial over A. Choosing a trivialization $h:p^{-1}(A)\to A\times\mathbb{C}^n$, let E/h be the quotient space of E under the identifications $h^{-1}(x,v)\sim h^{-1}(y,v)$ for $x,y\in A$. There is then an induced projection $E/h\to X/A$. To see that this is a vector bundle we need to find a local trivialization over a neighborhood of the point A/A.

We claim that E being trivial over A implies that it is trivial over some neighborhood of A. In many cases this holds because there is a neighborhood which deformation retracts onto A, so since $Vect^n_{\mathbb{C}}(A)$ depends only on the homotopy type of A, it follows that E must be trivial over this neighborhood. In the general case one can argue as follows. The local trivialization h over A determines sections $s_i: A \to E$ which form a basis in each fiber over A. Choose a cover of A by open sets U_j over which E is trivial. Via a local trivialization, each section s_i can be regarded as a map from $A \cap U_j$ to a single fiber, so by the Tietze extension theorem we obtain a section $s_{ij}: U_j \to E$ extending s_i. Then if $\{\varphi_j, \varphi\}$ is a partition of unity subordinate to the cover $\{U_j, X - A\}$ of X, the sum $\sum_j \varphi_j s_{ij}$ gives an extension of s_i to a section defined on all of X. Since these sections form a basis in each fiber over A, they must form a basis in all nearby fibers. Namely, over U_j the extended s_i's can be viewed as a square-matrix-valued function having nonzero determinant at each point of A, hence at nearby points as well.

Thus the local trivialization h over A extends to a local trivialization over a neighborhood U of A, and this induces a local trivialization of E/h over U/A, so E/h is a vector bundle. It remains only to verify that $E \approx q^*(E/h)$. We have a commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & E/h \\ \big\downarrow{\scriptstyle p} & & \big\downarrow \\ X & \xrightarrow{\;q\;} & X/A \end{array}$$

The quotient map $E \to E/h$ is an isomorphism on fibers, so this map and p give an isomorphism $E \approx q^*(E/h)$. $\square$

There is an easy way to extend the exact sequence $\widetilde{K}(X/A) \to \widetilde{K}(X) \to \widetilde{K}(A)$ to the left, using the following diagram, where C and S denote cone and suspension:

$$\begin{array}{ccccccccc} A & \hookrightarrow & X & \hookrightarrow & X \cup CA & \hookrightarrow & (X \cup CA) \cup CX & \hookrightarrow & \big((X \cup CA) \cup CX\big) \cup C(X \cup CA) \\ & & & & \big\downarrow \simeq & & \big\downarrow \simeq & & \big\downarrow \simeq \\ & & & & X/A & & SA & & SX \end{array}$$

In the first row, each space is obtained from its predecessor by attaching a cone on the subspace two steps back in the sequence. The vertical maps are the quotient maps obtained by collapsing the most recently attached cone to a point. In many cases the quotient map collapsing a contractible subspace to a point is a homotopy equivalence, hence induces an isomorphism on $\widetilde{K}$. This conclusion holds generally, in fact:

Lemma 2.4. *If A is contractible, the quotient map $q: X \to X/A$ induces a bijection $q^*: Vect^n(X/A) \to Vect^n(X)$ for all n.*

Proof: A vector bundle $E \to X$ must be trivial over A since A is contractible. A trivialization h gives a vector bundle $E/h \to X/A$ as in the proof of the previous proposition.

We assert that the isomorphism class of E/h does not depend on h. This can be seen as follows. Given two trivializations h_0 and h_1, by writing $h_1 = (h_1h_0^{-1})h_0$ we see that h_0 and h_1 differ by an element of $g_x \in GL_n(\mathbb{C})$ over each point $x \in A$. The resulting map $g : A \to GL_n(\mathbb{C})$ is homotopic to a constant map $x \mapsto \alpha \in GL_n(\mathbb{C})$ since A is contractible. Writing now $h_1 = (h_1h_0^{-1}\alpha^{-1})(\alpha h_0)$, we see that by composing h_0 with α in each fiber, which does not change E/h_0, we may assume that α is the identity. Then the homotopy from g to the identity gives a homotopy H from h_0 to h_1. In the same way that we constructed E/h we construct a vector bundle $(E \times I)/H \to (X/A) \times I$ restricting to E/h_0 over one end and E/h_1 over the other, hence $E/h_0 \approx E/h_1$.

Thus we have a well-defined map $Vect^n(X) \to Vect^n(X/A)$, $E \mapsto E/h$, and this is an inverse to q^* since $q^*(E/h) \approx E$ as we noted in the preceding proposition, and for a bundle $E \to X/A$ we have $q^*(E)/h \approx E$ for the evident trivialization h of $q^*(E)$ over A
$\square$

Thus the previous proposition implies that we have a long exact sequence of $\widetilde{K}$ groups

$$\cdots \to \widetilde{K}(SX) \to \widetilde{K}(SA) \to \widetilde{K}(X/A) \to \widetilde{K}(X) \to \widetilde{K}(A)$$

For example, if $X = A \vee B$ then $X/A = B$ and the sequence breaks up into split short exact sequences, which implies that the map $\widetilde{K}(X) \to \widetilde{K}(A) \oplus \widetilde{K}(B)$ obtained by restriction to A and B is an isomorphism.

We can use this exact sequence to obtain a version of the external product for $\widetilde{K}$, of the form $\otimes : \widetilde{K}(X) \otimes \widetilde{K}(Y) \to \widetilde{K}(X \wedge Y)$, where $X \wedge Y = X \times Y / X \vee Y$ and $X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y \subset X \times Y$ for chosen 0-cells $x_0 \in X$ and $y_0 \in Y$. To define this reduced product, consider the long exact sequence for the pair $(X \times Y, X \vee Y)$:

$$\begin{array}{ccccccccc}
\widetilde{K}(S(X\times Y)) & \to & \widetilde{K}(S(X\vee Y)) & \to & \widetilde{K}(X\wedge Y) & \to & \widetilde{K}(X\times Y) & \to & \widetilde{K}(X\vee Y) \\
 & & \wr\wr & & & & & & \wr\wr \\
 & & \widetilde{K}(SX)\oplus\widetilde{K}(SY) & & & & & & \widetilde{K}(X)\oplus\widetilde{K}(Y)
\end{array}$$

The second of the two vertical isomorphisms here was noted earlier, and the first vertical isomorphism arises in similar fashion since $S(X \vee Y) \simeq SX \vee SY$. The last horizontal map in the sequence is a split surjection, with splitting $\widetilde{K}(X) \oplus \widetilde{K}(Y) \to \widetilde{K}(X \times Y)$, $(a, b) \mapsto p_1^*(a) + p_2^*(b)$ where p_1 and p_2 are the projections of $X \times Y$ onto X and Y. Similarly, the first map splits via $(Sp_1)^* + (Sp_2)^*$. So we get a splitting $\widetilde{K}(X \times Y) \approx \widetilde{K}(X \wedge Y) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y)$.

If we are given elements $a \in \widetilde{K}(X) = Ker(K(X) \to K(x_0))$ and $b \in \widetilde{K}(Y) = Ker(K(Y) \to K(y_0))$, then the external product $a \otimes b = p_1^*(a)p_2^*(b) \in K(X \times Y)$ has

$p_1^*(a)$ restricting to zero in $K(Y)$ and $p_2^*(b)$ restricting to zero in $K(X)$, so $p_1^*(a)p_2^*(b)$ restricts to zero in both $K(X)$ and $K(Y)$, hence in $K(X \vee Y)$. In particular, $a \otimes b$ lies in $\widetilde{K}(X \times Y)$, and from the short exact sequence above, $a \otimes b$ pulls back to a unique element of $\widetilde{K}(X \wedge Y)$, also written $a \otimes b$. This defines the reduced external product $\otimes : \widetilde{K}(X) \otimes \widetilde{K}(Y) \to \widetilde{K}(X \wedge Y)$. The two forms of the external product are related by the diagram:

$$\begin{array}{ccccccccc}
K(X)\otimes K(Y) & \approx & (\widetilde{K}(X)\otimes\widetilde{K}(Y)) & \oplus & \widetilde{K}(X) & \oplus & \widetilde{K}(Y) & \oplus & \mathbb{Z} \\
\downarrow & & \downarrow & & \| & & \| & & \| \\
K(X \times Y) & \approx & \widetilde{K}(X \wedge Y) & \oplus & \widetilde{K}(X) & \oplus & \widetilde{K}(Y) & \oplus & \mathbb{Z}
\end{array}$$

Since $S^n \wedge X$ is the n-fold iterated reduced suspension $\Sigma^n X$, which is a quotient of the ordinary n-fold suspension $S^n X$ obtained by collapsing an n-disk in $S^n X$ to a point, the quotient map $S^n X \to S^n \wedge X$ induces an isomorphism on $\widetilde{K}$ by the previous lemma. Then the reduced external product gives a homomorphism $\beta : \widetilde{K}(X) \to \widetilde{K}(S^2 X)$, $\beta(a) = (H-1) \otimes a$, where H is the canonical line bundle over $S^2 = \mathbb{C}\mathrm{P}^1$. The version of Bott Periodicity for reduced K-theory states that this is an isomorphism. This is equivalent to the unreduced version by the preceding diagram.

As we saw earlier, a finite cell complex pair (X, A) gives rise to an exact sequence of $\widetilde{K}$ groups, the first row in the following diagram.

$$\begin{array}{ccccccccccccccc}
\widetilde{K}(S^2X) & \to & K(S^2A) & \to & \widetilde{K}(S(X/A)) & \to & \widetilde{K}(SX) & \to & \widetilde{K}(SA) & \to & \widetilde{K}(X/A) & \to & \widetilde{K}(X) & \to & \widetilde{K}(A) \\
\| & & \| & & \| & & \| & & \| & & \| & & \| & & \| \\
\widetilde{K}^{-2}(X) & \to & \widetilde{K}^{-2}(A) & \to & \widetilde{K}^{-1}(X,A) & \to & \widetilde{K}^{-1}(X) & \to & \widetilde{K}^{-1}(A) & \to & \widetilde{K}^{0}(X,A) & \to & \widetilde{K}^{0}(X) & \to & \widetilde{K}^{0}(A) \\
\uparrow\approx & & \uparrow\approx & & & & & & & & & & & & \\
\widetilde{K}^{0}(X) & \longrightarrow & \widetilde{K}^{0}(A) & & & & & & & & & & & &
\end{array}$$

Defining $\widetilde{K}^{-n}(X) = \widetilde{K}(S^n X)$ and $\widetilde{K}^{-n}(X,A) = \widetilde{K}(S^n(X/A))$, this sequence can be written as in the second row. Negative indices are chosen here so that the "coboundary" map in this sequence increases dimension, as in ordinary cohomology. The lower left corner of the diagram contains the Bott periodicity isomorphisms β. This square commutes since external tensor product with $H-1$ commutes with maps between spaces. So the lower long exact sequence can be rolled up into a six-term periodic exact sequence. It is reasonable to extend the definition of $\widetilde{K}^n$ to positive n via periodicity, and then the long exact sequence can be written:

$$\begin{array}{ccccc}
\widetilde{K}^0(X,A) & \longrightarrow & \widetilde{K}^0(X) & \longrightarrow & \widetilde{K}^0(A) \\
\uparrow & & & & \downarrow \\
\widetilde{K}^1(A) & \longleftarrow & \widetilde{K}^1(X) & \longleftarrow & \widetilde{K}^1(X,A)
\end{array}$$

Replacing X and Y by S^iX and S^jY in the external product $\widetilde{K}(X)\otimes\widetilde{K}(Y) \to \widetilde{K}(X\wedge Y)$, we obtain a product $\widetilde{K}^i(X)\otimes\widetilde{K}^j(Y)\to\widetilde{K}^{i+j}(X\wedge Y)$. If we define $\widetilde{K}^*(X) = \widetilde{K}^0(X)\oplus\widetilde{K}^1(X)$, then this gives a product $\widetilde{K}^*(X)\otimes\widetilde{K}^*(Y)\to\widetilde{K}^*(X\wedge Y)$. The relative form of this is a product $\widetilde{K}^*(X,A)\otimes\widetilde{K}^*(Y,B)\to\widetilde{K}^*(X\times Y, X\times B\cup A\times Y)$, coming from the products $\widetilde{K}(\Sigma^i(X/A))\otimes\widetilde{K}(\Sigma^j(Y/B)) \longrightarrow \widetilde{K}(\Sigma^{i+j}(X/A\,\wedge\,Y/B))$ using the natural identification $(X\times Y)/(X\times B\cup A\times Y) = X/A\,\wedge\,Y/B$.

If we compose the product $\widetilde{K}^*(X)\otimes\widetilde{K}^*(X)\to\widetilde{K}^*(X\wedge X)$ with the map $\widetilde{K}^*(X\wedge X)\to\widetilde{K}^*(X)$ induced by the diagonal map $X\to X\wedge X$, $x\mapsto(x,x)$, then we obtain a multiplication on $\widetilde{K}^*(X)$ making it into a ring. The general relative form of this is a product $\widetilde{K}^*(X,A)\otimes\widetilde{K}^*(X,B)\to\widetilde{K}(X,A\cup B)$ which is induced by the relativized diagonal map $X/(A\cup B)\to X/A\,\wedge\,Y/B$.

Example. Suppose that $X = A\cup B$ with both A and B contractible, for example if X is a suspension and A and B are its two cones. Then the product $\widetilde{K}^*(X)\otimes\widetilde{K}^*(X)\to\widetilde{K}(X)$ is identically zero since it is equivalent to the composition $\widetilde{K}^*(X,A)\otimes\widetilde{K}^*(X,B)\to\widetilde{K}(X,A\cup B)\to\widetilde{K}(X)$ and $\widetilde{K}(X,A\cup B)=0$ since $X=A\cup B$. Thus the product in $\widetilde{K}^*(S^n)\approx\mathbb{Z}$ is trivial.

For a pair (X,A) the diagonal map $X\to X\wedge X$ induces a well-defined map $X/A\to X\wedge X/A$ which leads to a product $\widetilde{K}^*(X)\otimes\widetilde{K}^*(X,A)\to\widetilde{K}^*(X,A)$ making $\widetilde{K}^*(X,A)$ a module over $\widetilde{K}^*(X)$. In fact:

Proposition 2.5. *The exact sequence*

$$\begin{tikzcd} \widetilde{K}^*(X,A) \arrow[rr] & & \widetilde{K}^*(X) \arrow[dl] \\ & \widetilde{K}^*(A) \arrow[ul] & \end{tikzcd}$$

is an exact sequence of $\widetilde{K}^*(X)$-modules, with the maps $\widetilde{K}^*(X)$-module homomorphisms.

Proof: The $\widetilde{K}^*(X)$-module structure on $\widetilde{K}^*(A)$ is defined by $\xi\cdot\alpha = i^*(\xi)\alpha$ where i is the inclusion $A\hookrightarrow X$ and the product on the right side of the equation is multiplication in the ring $\widetilde{K}^*(A)$. To see that the maps in the exact sequence are module homomorphisms we look at the diagram

$$\begin{tikzcd} \widetilde{K}(S^jSA) \arrow[r] \arrow[d] & \widetilde{K}(S^j(X/A)) \arrow[r] \arrow[d] & \widetilde{K}(S^jX) \arrow[r] \arrow[d] & \widetilde{K}(S^jA) \arrow[d] \\ \widetilde{K}(S^iX\wedge S^jSA) \arrow[r] \arrow[d] & \widetilde{K}(S^iX\wedge S^j(X/A)) \arrow[r] \arrow[d] & \widetilde{K}(S^iX\wedge S^jX) \arrow[r] \arrow[d] & \widetilde{K}(S^iX\wedge S^jA) \arrow[d] \\ \widetilde{K}(S^{i+j}SA) \arrow[r] & \widetilde{K}(S^{i+j}(X/A)) \arrow[r] & \widetilde{K}(S^{i+j}X) \arrow[r] & \widetilde{K}(S^{i+j}A) \end{tikzcd}$$

where the vertical maps between the first two rows are external product with a fixed element of $\widetilde{K}(S^iX)$ and the vertical maps between the second and third rows are induced by diagonal maps. What we must show is that the diagram commutes. For the upper two rows this follows from naturality of external product since the horizontal maps are induced by maps between spaces. The lower two rows are induced from suspensions of maps between spaces,

$$\begin{tikzcd}
X \wedge SA & X \wedge X/A \arrow[l] & X \wedge X \arrow[l] & X \wedge A \arrow[l] \\
SA \arrow[u] & X/A \arrow[l] \arrow[u] & X \arrow[l] \arrow[u] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

so it suffices to show this diagram commutes up to homotopy. This is obvious for the middle and right squares. The left square can be rewritten

$$\begin{tikzcd}
X \wedge SA & X \wedge X \cup CA \arrow[l] \\
SA \arrow[u] & X \cup CA \arrow[l] \arrow[u]
\end{tikzcd}$$

where the horizontal maps collapse the copy of X in $X \cup CA$ to a point, the left vertical map sends $(a,s) \in SA$ to $(a,a,s) \in X \wedge SA$, and the right vertical map sends $x \in X$ to $(x,x) \in X \cup CA$ and $(a,s) \in CA$ to $(a,a,s) \in X \wedge CA$. Commutativity is then obvious. $\square$

Whereas multiplication in $\widetilde{K}(X)$ is commutative, in $\widetilde{K}^*(X)$ this is only true up to sign:

Proposition 2.6. *$\alpha\beta = (-1)^{ij}\beta\alpha$ for $\alpha \in \widetilde{K}^i(X)$ and $\beta \in \widetilde{K}^j(X)$.*

Proof: The product is the composition

$$\widetilde{K}(S^i \wedge X) \otimes \widetilde{K}(S^j \wedge X) \longrightarrow \widetilde{K}(S^i \wedge S^j \wedge X \wedge X) \longrightarrow \widetilde{K}(S^i \wedge S^j \wedge X)$$

where the first map is external product and the second is induced by the diagonal map on the X factors. Replacing the product $\alpha\beta$ by the product $\beta\alpha$ amounts to switching the two factors in the first term $\widetilde{K}(S^i \wedge X) \otimes \widetilde{K}(S^j \wedge X)$, and this corresponds to switching the S^i and S^j factors in the third term $\widetilde{K}(S^i \wedge S^j \wedge X)$. Viewing $S^i \wedge S^j$ as the smash product of $i+j$ copies of S^1, then switching S^i and S^j in $S^i \wedge S^j$ is a product of ij transpositions of adjacent factors. Transposing the two factors of $S^1 \wedge S^1$ is equivalent to reflection of S^2 across an equator. Thus it suffices to see that switching the two ends of a suspension SY induces multiplication by -1 in $\widetilde{K}(SY)$. If we view $\widetilde{K}(SY)$ as $\langle Y, U\rangle$, then switching ends of SY corresponds to the map $U \to U$ sending a matrix to its inverse. We noted in the proof of Proposition 2.2 that the group operation in $K(SY)$ is the same as the operation induced by the product in U, so the result follows. $\square$

It is often convenient to have an unreduced version of the groups $\widetilde{K}^n(X)$, and this can easily be done by the simple device of defining $K^n(X)$ to be $\widetilde{K}^n(X_+)$ where X_+ is X with a disjoint basepoint $+$ adjoined. For $n = 0$ this is consistent with the relation between K and $\widetilde{K}$ since $K^0(X) = \widetilde{K}^0(X_+) = \widetilde{K}(X_+) = Ker(K(X_+) \to K(+)) = K(X)$. For $n = 1$ this definition yields $K^1(X) = \widetilde{K}^1(X)$ since $S(X_+) \simeq SX \vee S^1$ and $\widetilde{K}(SX \vee S^1) \approx \widetilde{K}(SX) \oplus \widetilde{K}(S^1) \approx \widetilde{K}(SX)$ since $\widetilde{K}(S^1) = 0$. For a pair (X, A) with $A \neq \varnothing$ one defines $K^n(X, A) = \widetilde{K}^n(X, A)$, and then the six-term long exact sequence is valid also for unreduced K-groups. When $A = \varnothing$ this remains valid if we interpret $X/A = X/\varnothing$ as X_+.

Since $X_+ \wedge Y_+ = (X \times Y)_+$, the external product $\widetilde{K}^*(X) \otimes \widetilde{K}^*(Y) \to \widetilde{K}^*(X \wedge Y)$. gives a product $K^*(X) \otimes K^*(Y) \to K^*(X \times Y)$. Taking $X = Y$ and composing with the map $K^*(X \times X) \to K^*(X)$ induced by the diagonal map $X \to X \times X$, $x \mapsto (x, x)$, we get a product $K^*(X) \otimes K^*(X) \to K^*(X)$ which makes $K^*(X)$ into a ring.

In the relative case we have a product $K^i(X, A) \otimes K^j(Y, B) \to K^{i+j}(X \times Y, X \times B \cup A \times Y)$, which is by definition the reduced external product $\widetilde{K}(\Sigma^i(X/A)) \otimes \widetilde{K}(\Sigma^j(Y/B)) \longrightarrow \widetilde{K}(\Sigma^{i+j}(X/A \wedge Y/B))$, using the natural identification $(X \times Y)/(X \times B \cup A \times Y) = X/A \wedge Y/B)$. This works when $A = \varnothing$ since we interpret $X/A = X/\varnothing$ as X_+, and similarly if $Y = \varnothing$. Via the diagonal map we obtain also a product $K^i(X, A) \otimes K^j(X, B) \to K^{i+j}(X, A \cup B)$.

With these definitions the preceding two propositions hold equally well for unreduced K-groups.

2. Bott Periodicity

The form of the Bott periodicity theorem which we shall prove is the assertion that the external product map $\mu : K(X) \otimes K(S^2) \to K(X \times S^2)$ is an isomorphism for all compact Hausdorff spaces X. The present section will be devoted entirely to the proof of this theorem. Nothing in the proof will be used elsewhere in the book except in the proof of Bott periodicity for real K-theory in Chapter 4, so the reader who is willing to accept the theorem on faith may proceed to 2.3 without any loss of continuity.

The main work in proving the theorem will be to prove the surjectivity of μ. Injectivity will then be proved by a closer examination of the surjectivity argument. The proof of surjectivity will involve taking an arbitrary vector bundle over $X \times S^2$ and performing various simplifications and modifications of it until it is clearly in the image of μ.

Clutching Functions

Given a vector bundle $p : E \to X$, let $f : E \times S^1 \to E \times S^1$ be an automorphism of the product bundle $p \times 1\!\!1 : E \times S^1 \to X \times S^1$. Thus for each $x \in X$ and $z \in S^1$, f specifies an isomorphism $f(x,z)$ of the fiber $p^{-1}(x)$. From E and f we construct a vector bundle over $X \times S^2$ by taking two copies of $E \times D^2$ and identifying the subspaces $E \times S^1$ via f. We write this bundle as $[E, f]$, and call f a *clutching function* for $[E, f]$. If $f_t : E \times S^1 \to E \times S^1$ is a homotopy of clutching functions, then $[E, f_0] \approx [E, f_1]$ since from the homotopy f_t we can construct a vector bundle over $X \times S^2 \times I$ restricting to $[E, f_0]$ and $[E, f_1]$ over $X \times S^2 \times \{0\}$ and $X \times S^2 \times \{1\}$. From the definitions it is a routine exercise to verify the formula $[E_1, f_1] \otimes [E_2, f_2] \approx [E_1 \otimes E_2, f_1 \otimes f_2]$.

Here are some examples:

1. $[E, 1\!\!1]$ is the external product $E \otimes 1 = \mu(E, 1)$.

2. Taking X to be a point, then $[1, z] \approx H$ where "1" is the trivial line bundle over X and "z" means scalar multiplication by $z \in S^1 \subset \mathbb{C}$. More generally, $[1, z^n] \approx H^n$ since $z^i \otimes z^j = z^{i+j}$. The formula $[1, z^n] \approx H^n$ holds also for negative n if we define $H^{-1} = [1, z^{-1}]$, so that $H \otimes H^{-1} \approx 1$.

3. $[E, z^n f] \approx [E \otimes 1, f \otimes z^n] \approx [E, f] \otimes [1, z^n] \approx [E, f] \otimes H^n$ where "H^n" here is an abbreviation for the pullback of H^n via the projection $X \times S^2 \to S^2$, in keeping with the notation for external tensor product.

4. $[E, z^n] \approx E \otimes H^n = \mu(E, H^n)$, a special case of 3.

Every vector bundle $E' \to X \times S^2$ is isomorphic to $[E, f]$ for some E and f. Namely, let $S^1 \subset \mathbb{C} \cup \{\infty\} = S^2$ decompose S^2 into the two disks D_0 and D_∞, with E_α the restriction of E' over $X \times D_\alpha$ and E the restriction of E' over $X \times \{1\}$. Pulling back a deformation retraction of $X \times D_\alpha$ onto $X \times \{1\}$ gives an isomorphism $h_\alpha : E_\alpha \to E \times D_\alpha$, and $f = h_0 h^{-1}$ is a clutching function for E'. We may assume f is normalized to be the identity over $X \times \{1\}$ since we may normalize an isomorphism $h_\alpha : E_\alpha \to E \times D_\alpha$ by composing it over each $X \times \{z\}$ with the inverse of its restriction over $X \times \{1\}$. Any two choices of normalized h_α are homotopic through normalized h_α's since they differ by a map g_α from D_α to the automorphisms of E, with $g_\alpha(1) = \mathbb{1}$, and such a g_α is homotopic to the constant map $\mathbb{1}$ by composing it with a deformation retraction of D_α to 1. Thus any two choices f_0 and f_1 of normalized clutching functions f for E' are joined by a homotopy of normalized clutching functions f_t.

The idea of the proof is to reduce from arbitrary clutching functions to successively simpler clutching functions. The first step will be to reduce to *Laurent polynomial* clutching functions, which have the form $\ell(x,z) = \sum_{|i| \le n} a_i(x) z^i$ where $a_i : E \to E$ restricts to a linear transformation $a_i(x)$ in each fiber $p^{-1}(x)$. We call such an a_i an *endomorphism* of E since the linear transformations $a_i(x)$ need not be invertible, though their linear combination $\sum a_i(x) z^i$ must be invertible since clutching functions are automorphisms.

Proposition 2.7. *Every vector bundle $[E, f]$ is isomorphic to $[E, \ell]$ for some Laurent polynomial clutching function ℓ. Laurent polynomial clutching functions ℓ_0 and ℓ_1 which are homotopic through clutching functions are homotopic by a Laurent polynomial clutching function homotopy $\ell_t(x,z) = \sum a_i(x,t) z^i$.*

Before beginning the proof we need a lemma. For a compact space X we wish to approximate a continuous function $f : X \times S^1 \to \mathbb{C}$ by Laurent polynomial functions $\sum_{|n| \le N} a_n(x) z^n = \sum_{|n| \le N} a_n(x) e^{in\theta}$, where each a_n is a continuous function $X \to \mathbb{C}$. Motivated by Fourier series, we set:

$$a_n(x) = \frac{1}{2\pi} \int_{S^1} f(x,\theta) e^{-in\theta} \, d\theta$$

For positive real r let $u(x,r,\theta) = \sum_{n \in \mathbb{Z}} a_n(x) r^{|n|} e^{in\theta}$. For fixed $r < 1$, this series converges absolutely and uniformly as (x,θ) ranges over $X \times S^1$, by comparison with the geometric series $\sum r^n$, since $|f(x,\theta)|$ bounded implies $|a_n(x)|$ bounded. If we can show that $u(x,r,\theta)$ approaches $f(x,\theta)$ uniformly in x and θ as r goes to 1, then sums of finitely many terms in the series for $u(x,r,\theta)$ for r near 1 will give the desired approximations to f by Laurent polynomial functions.

Lemma 2.8. *As $r \to 1$, $u(x,r,\theta) \to f(x,\theta)$ uniformly in x and θ.*

Proof: For $r < 1$ we have

$$u(x,r,\theta) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{S^1} r^{|n|} e^{in(\theta-t)} f(x,t)\, dt$$
$$= \int_{S^1} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta-t)} f(x,t)\, dt$$

where the order of summation and integration can be interchanged since the series in the latter formula converges uniformly, by comparison with the geometric series $\sum_n r^n$. Define the Poisson kernel

$$P(r,\varphi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\varphi}$$

so that $u(x,r,\theta) = \int_{S^1} P(r,\theta-t) f(x,t)\, dt$. By summing the two geometric series for positive and negative n in the formula for $P(r,\varphi)$, one computes that

$$P(r,\varphi) = \frac{1}{2\pi} \cdot \frac{1-r^2}{1-2r\cos\varphi + r^2}$$

Three basic facts about $P(r,\varphi)$ which we shall need are:

(a) As a function of φ, $P(r,\varphi)$ is even, of period 2π, and monotone decreasing on $[0,\pi]$, since the same is true of $\cos\varphi$ which appears in the denominator of $P(r,\varphi)$ with a minus sign.

(b) $\int_{S^1} P(r,\varphi)\, d\varphi = 1$ for $r < 1$, as one sees by integrating the series for $P(r,\varphi)$ term by term.

(c) For fixed $\varphi \in (0,\pi)$, $P(r,\varphi) \to 0$ as $r \to 1$ since the numerator of $P(r,\varphi)$ approaches 0 and the denominator approaches $2 - 2\cos\varphi \neq 0$.

Now to show uniform convergence of $u(x,r,\theta)$ to $f(x,\theta)$ we first observe that, using (b), we have:

$$\big|u(x,r,\theta) - f(x,\theta)\big| = \Big| \int_{S^1} P(r,\theta-t) f(x,t)\, dt - \int_{S^1} P(r,\theta-t) f(x,\theta)\, dt \Big|$$
$$\leq \int_{S^1} P(r,\theta-t) \big|f(x,t) - f(x,\theta)\big|\, dt$$

Given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x,t) - f(x,\theta)| < \varepsilon$ for $|t-\theta| < \delta$ and all x, since f is uniformly continuous on the compact space $X \times S^1$. Let I_δ denote the integral $\int P(r,\theta-t)\, |f(x,t) - f(x,\theta)|\, dt$ over the interval $|t-\theta| \leq \delta$ and let I'_δ denote this integral over the rest of S^1. Then we have:

$$I_\delta \leq \int_{|t-\theta|\leq\delta} P(r,\theta-t)\, \varepsilon\, dt \leq \varepsilon \int_{S^1} P(r,\theta-t)\, dt = \varepsilon$$

By (a) the maximum value of $P(r,\theta-t)$ on $|t-\theta|\geq\delta$ is $P(r,\delta)$. So:

$$I'_\delta \leq P(r,\delta)\int_{S^1}|f(x,t)-f(x,\theta)|\,dt$$

The integral here has a uniform bound for all x and θ since f is bounded. Thus by (c) we can make $I'_\delta\leq\varepsilon$ by taking r close enough to 1. Therefore $|u(x,r,\theta)-f(x,\theta)|\leq I_\delta+I'_\delta\leq 2\varepsilon$. □

Proof of Proposition 2.7: Choosing a Hermitian inner product on E, the endomorphisms of $E\times S^1$ form a vector space $End(E\times S^1)$ with a norm $\|\alpha\| = sup_{|v|=1}|\alpha(v)|$. The triangle inequality holds for this norm, so balls in $End(E\times S^1)$ are convex. The subspace $Aut(E\times S^1)$ of automorphisms is open in the topology defined by this norm since it is the preimage of $(0,\infty)$ under the continuous map $End(E\times S^1)\to[0,\infty)$, $\alpha\mapsto inf_{(x,z)\in X\times S^1}|det(\alpha(x,z))|$. Thus to prove the first statement of the lemma it will suffice to show that Laurent polynomials are dense in $End(E\times S^1)$, since a sufficiently close Laurent polynomial approximation ℓ to f will then be homotopic to f via the linear homotopy $t\ell+(1-t)f$ through clutching functions. The second statement follows similarly by approximating a homotopy from ℓ_0 to ℓ_1, viewed as an automorphism of $E\times S^1\times I$, by a Laurent polynomial homotopy ℓ'_t, then combining this with linear homotopies from ℓ_0 to ℓ'_0 and ℓ_1 to ℓ'_1 to obtain a homotopy ℓ_t from ℓ_0 to ℓ_1.

To show that every $f\in End(E\times S^1)$ can be approximated by Laurent polynomial endomorphisms, first choose open sets U_i covering X together with isomorphisms $h_i : p^{-1}(U_i)\to U_i\times\mathbb{C}^{n_i}$. We may assume h_i takes the chosen inner product in $p^{-1}(U_i)$ to the standard inner product in $\mathbb{C}^{n_i}$, by applying the Gram-Schmidt process to h_i^{-1} of the standard basis vectors. Let $\{\varphi_i\}$ be a partition of unity subordinate to $\{U_i\}$ and let X_i be the support of φ_i, a compact set in U_i. Via h_i, the linear maps $f(x,z)$ for $x\in X_i$ can be viewed as matrices. The entries of these matrices define functions $X_i\times S^1\to\mathbb{C}$. By the lemma we can find Laurent polynomial matrices $\ell_i(x,z)$ whose entries uniformly approximate those of $f(x,z)$ for $x\in X_i$. It follows easily that ℓ_i approximates f in the $\|\cdot\|$ norm. From the Laurent polynomial approximations ℓ_i over X_i we form the convex linear combination $\ell=\sum\varphi_i\ell_i$, a Laurent polynomial approximating f over all of $X\times S^1$. □

A Laurent polynomial clutching function can be written $\ell = z^{-m}q$ for a polynomial clutching function q, and then we have $[E,\ell]\approx[E,q]\otimes H^{-m}$. The next step is to reduce polynomial clutching functions to linear clutching functions.

Proposition 2.9. *If q is a polynomial clutching function of degree at most n, then $[E,q]\oplus[nE,\mathbb{1}]\approx[(n+1)E,L^n q]$ for a linear clutching function $L^n q$.*

Proof: Let $q(x,z)=a_n(x)z^n+\cdots+a_0(x)$, and consider the matrices

$$\begin{pmatrix} 1 & -z & 0 & \cdots & 0 & 0 \\ 0 & 1 & -z & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -z \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & q \end{pmatrix}$$

which define endomorphisms of $(n+1)E$. We can pass from the first matrix to the second by a sequence of elementary row and column operations in the following way. In the first matrix, add z times the first column to the second column, then z times the second column to the third, etc. This produces all 0's above the diagonal, and the polynomial q in the lower right corner. Then for each i, subtract the appropriate multiple of the i^{th} row from the last row.

The second matrix above is a clutching function for $(nE\otimes 1)\oplus[E,q]$. The first matrix has the same determinant as the second matrix, hence is also invertible, and is therefore a clutching function for $(n+1)E$. Denote this clutching function by $L^n q$. Since $L^n q$ has the form $A(x)z+B(x)$ for matrices $A(x)$ and $B(x)$, it is a linear clutching function. The two matrices displayed above define homotopic clutching functions since the elementary row and column operations can be achieved by continuous one-parameter families of such operations, e.g., add tz times the first column to the second, etc. Hence $[E,q]\oplus(nE\otimes 1)\approx[(n+1)E,L^n q]$. □

Linear Clutching Functions

For linear clutching functions $a(x)z+b(x)$ we have the following key fact:

Proposition 2.10. *Given a bundle $[E,a(x)z+b(x)]$, there is a splitting $E\approx E_+\oplus E_-$ with $[E,a(x)z+b(x)]\approx[E_+,\mathbb{1}]\oplus[E_-,z]$.*

Proof: The first step is to reduce to the case that $a(x)$ is the identity for all x. Consider the expression:

$$(*)\qquad (1+tz)a(x)\frac{z+t}{1+tz}+b(x)=a(x)+tb(x)z+ta(x)+b(x)$$

When $t=0$ this equals $a(x)z+b(x)$. For $0\le t<1$, $(*)$ defines an invertible linear transformation since the left-hand side is obtained from $a(x)z+b(x)$ by first applying the substitution $z \mapsto (z+t)/(1+tz)$ which takes S^1 to itself, then multiplying by the nonzero scalar $1+tz$. Therefore $(*)$ defines a homotopy of clutching functions as t goes from 0 to $t_0 < 1$. In the right-hand side of $(*)$ the term $a(x)+tb(x)$ is invertible for $t=1$ since it is the restriction of $a(x)z+b(x)$ to $z=1$. Therefore $a(x)+tb(x)$ is invertible for $t=t_0$ near 1, as the continuous function $t \mapsto inf_{x\in X}|det(a(x)+tb(x))|$ is nonzero for $t=1$, hence also for t near 1. Now we use the simple fact that $[E,fg] \approx [E,f]$ for any isomorphism $g:E\to E$. This allows us to replace the clutching function on the right-hand side of $(*)$ by the clutching function $z+[t_0a(x)+b(x)][a(x)+t_0b(x)]^{-1}$, reducing to the case of clutching functions of the form $z+b(x)$.

Since $z+b(x)$ is invertible for all x, $b(x)$ has no eigenvalues on the unit circle S^1.

Lemma 2.11. *Let $b:E\to E$ be an endomorphism having no eigenvalues on the unit circle S^1. Then there are unique subbundles E_+ and E_- of E such that:*

1. $E = E_+ \oplus E_-$.
2. $b(E_\pm) \subset E_\pm$
3. *$b\big|E_+$ has all its eigenvalues outside S^1 and $b\big|E_-$ has all its eigenvalues inside S^1.*

Proof: Consider first the algebraic situation of a linear transformation $T:V\to V$ with characteristic polynomial $q(t)\in\mathbb{C}[t]$. Assuming $q(t)$ has no roots on S^1, factor $q(t)$ as $q_+(t)q_-(t)$ where $q_+(t)$ has all its roots outside S^1 and $q_-(t)$ has all its roots inside S^1. Define subspaces $V_\pm = Ker q_\pm(T):V\to V$. Since q_+ and q_- are relatively prime in $\mathbb{C}[t]$, there are polynomials r and s with $rq_+ + sq_- = 1$. From $q_+(T)q_-(T) = q(T) = 0$, we have $Im q_- \subset Ker q_+$, and the opposite inclusion follows from $r(T)q_+(T) + q_-(T)s(T) = \mathbb{1}$. Thus $Ker q_+ = Im q_-$, and similarly $Ker q_- = Im q_+$. From $q_+(T)r(T) + q_-(T)s(T) = \mathbb{1}$ we see that $Im q_+ + Im q_- = V$, and from $r(T)q_+(T) + s(T)q_-(T) = \mathbb{1}$ we obtain $Ker q_+ \cap Ker q_- = 0$. Hence $V = V_+ \oplus V_-$. We have $T(V_\pm)\subset V_\pm$ since $q_\pm(T)(v) = 0$ implies $q_\pm(T)(T(v)) = T(q_\pm(T)(v)) = 0$. All eigenvalues of $T\big|V_\pm$ are roots of $q_\pm$ since $q_\pm(T) = 0$ on $V_\pm$. Thus conditions 1-3 hold for V_+ and V_-.

To see the uniqueness of V_+ and V_- satisfying 1-3, let $q'_\pm$ be the characteristic polynomial of $T\big|V_\pm$, so $q = q'_+q'_-$. All the linear factors of $q'_\pm$ must be factors of $q_\pm$ by condition 3, so the factorizations $q = q'_+q'_-$ and $q = q_+q_-$ must coincide up to scalar factors. Since $q'_\pm(T)$ is identically zero on $V_\pm$, so must be $q_\pm(T)$, hence $V_\pm \subset Ker q_\pm(T)$. Since $V = V_+\oplus V_-$ and $V = Ker q_+(T) \oplus Ker q_-(T)$, we must have $V_\pm = Ker q_\pm(T)$. This establishes the uniqueness of $V_\pm$.

As T varies continuously through linear transformations without eigenvalues on S^1, its characteristic polynomial $q(t)$ varies continuously through polynomials without roots in S^1. In this situation we assert that the factors $q_\pm$ of q vary continuously with q, assuming that q, q_+, and q_- are normalized to be monic polynomials. To see this we shall use the fact that for any circle C in $\mathbb{C}$ disjoint from the roots of q, the number of roots of q inside C, counted with multiplicity, equals the degree of the map $\gamma : C \to S^1$, $\gamma(z) = q(z)/|q(z)|$. To prove this fact it suffices to consider the case of a small circle C about a root $z = a$ of multiplicity m, so $q(t) = p(t)(t-a)^m$ with $p(a) \neq 0$. The homotopy

$$\gamma_s(z) = \frac{p(sa + (1-s)z)(z-a)^m}{|p(sa + (1-s)z)(z-a)^m\,|}$$

reduces one to the case $(t-a)^m$, where it is clear that the degree is m.

Thus for a small circle C about a root $z = a$ of q of multiplicity m, small perturbations of q produce polynomials q' which also have m roots $a_1, \cdots, a_m$ inside C, so the factor $(z-a)^m$ of q becomes a factor $(z-a_1)\cdots(z-a_m)$ of the nearby q'. Since the a_i's are near a, these factors of q and q' are close, and so $q'_\pm$ is close to $q_\pm$.

Next we observe that as T varies continuously through transformations without eigenvalues in S^1, the splitting $V = V_+ \oplus V_-$ also varies continuously. For this, recall that $V_+ = Im q_-(T)$ and $V_- = Im q_+(T)$. Choose a basis $v_1, \cdots, v_n$ for V so that $q_-(T)(v_1), \cdots, q_-(T)(v_k)$ is a basis for V_+ and $q_+(T)(v_{k+1}), \cdots, q_+(T)(v_n)$ is a basis for V_-. For nearby T these vectors vary continuously, hence remain independent. Thus the splitting $V = Im q_- \oplus Im q_+$ continues to hold for nearby T, and so the splitting $V = V_+ \oplus V_-$ varies continuously with T.

It follows that the union $E_\pm$ of the subspaces $V_\pm$ in all the fibers V of E is a subbundle, and so the proof of the lemma is complete. □

To finish the proof of Proposition 2.10, note that the lemma gives a splitting $[E, z + b(x)] \approx [E_+, z + b_+(x)] \oplus [E_-, z + b_-(x)]$ where b_+ and b_- are the restrictions of b. Since $b_+(x)$ has all its eigenvalues outside S^1, the formula $tz + b_+(x)$, $0 \le t \le 1$, defines a homotopy of clutching functions from $z + b_+(x)$ to $b_+(x)$. Hence $[E_+, z + b_+(x)] \approx [E_+, b_+(x)] \approx [E_+, \mathbb{1}]$. Similarly, $z + tb_-(x)$ defines a homotopy of clutching functions from $z + b_-(x)$ to z, so $[E_-, z + b_-(x)] \approx [E_-, z]$. □

For future reference we note that the splitting $[E, az + b] \approx [E_+, \mathbb{1}] \oplus [E_-, z]$ constructed in the proof of Proposition 2.10 preserves direct sums, namely, the splitting for a sum $[E_1 \oplus E_2, (a_1 z + b_1) \oplus (a_2 z + b_2)]$ has $(E_1 \oplus E_2)_\pm = (E_1)_\pm \oplus (E_2)_\pm$. This is because the first step of reducing to the case $a = \mathbb{1}$ clearly respects sums, and the uniqueness of the $\pm$-splitting in Lemma 2.11 guarantees that it preserves sums.

Conclusion of the Proof

The preceding propositions imply that in $K(X\times S^2)$ we have:

$$
\begin{aligned}
[E,f]=[E,z^{-m}q]&=[E,q]\otimes H^{-m}\\
&=[(n+1)E,L^nq]\otimes H^{-m}-[nE,\mathbb{1}]\otimes H^{-m}\\
&=[((n+1)E)_+,\mathbb{1}]\otimes H^{-m}+[((n+1)E)_-,z]\otimes H^{-m}-[nE,\mathbb{1}]\otimes H^{-m}\\
&=((n+1)E)_+\otimes H^{-m}+((n+1)E)_-\otimes H^{1-m}-nE\otimes H^{-m}
\end{aligned}
$$

This last expression is in the image of $\mu: K(X)\otimes K(S^2)\to K(X\times S^2)$. Since every vector bundle over $X\times S^2$ has the form $[E,f]$, it follows that μ is surjective.

To show μ is injective we shall construct $\nu: K(X\times S^2)\to K(X)\otimes K(S^2)$ such that $\nu\mu=\mathbb{1}$. To define ν the rough idea is to find some linear combination of terms $E\otimes H^k$ and $((n+1)E)_\pm\otimes H^k$ as in the preceding displayed formula, which is independent of all choices and satisfies $\nu\mu=\mathbb{1}$.

To investigate the dependence on m and n, we first derive the following two formulas, where $deg q\le n$:

(1) $[(n+2)E,L^{n+1}q]\approx[(n+1)E,L^nq]\oplus[E,\mathbb{1}]$

(2) $[(n+2)E,L^{n+1}(zq)]\approx[(n+1)E,L^nq]\oplus[E,z]$

The matrix representations of $L^{n+1}q$ and $L^{n+1}(zq)$ are:

$$
\begin{pmatrix}
1 & -z & 0 & \cdots & 0\\
0 & 1 & -z & \cdots & 0\\
\vdots & \vdots & \vdots & & \vdots\\
0 & 0 & 0 & 1 & -z\\
0 & a_n & a_{n-1} & \cdots & a_0
\end{pmatrix}
\qquad
\begin{pmatrix}
1 & -z & 0 & \cdots & 0 & 0\\
0 & 1 & -z & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & & \vdots & \vdots\\
0 & 0 & 0 & \cdots & 1 & -z\\
a_n & a_{n-1} & a_{n-2} & \cdots & a_0 & 0
\end{pmatrix}
$$

In the first matrix we can add z times the first column to the second column to eliminate the $-z$ in the first row, and then the first row and column give the summand $[E,\mathbb{1}]$ while the rest of the matrix gives $[(n+1)E,L^nq]$. Similarly, in the second matrix we add z^{-1} times the last column to the next-to-last column to make the $-z$ in the last column have all zeros in its row and column, which gives the splitting in (2) since $[E,-z]\approx[E,z]$, the clutching function $-z$ being the composition of the clutching function z with the automorphism -1 of E.

In view of the appearance of the correction terms $[E,\mathbb{1}]$ and $[E,z]$ in (1) and (2), it will be useful to know the "$\pm$" splittings for these two bundles:

(3) $[E_-,\mathbb{1}]=0$, hence $[E_+,\mathbb{1}]=[E,\mathbb{1}]$

(4) $[E_+, z] = 0$, hence $[E_-, z] = [E, z]$

Statement (4) is obvious from the definitions since the clutching function z is already in the form $z+b(x)$ with $b(x)=0$, so 0 is the only eigenvalue of $b(x)$ and hence $[E_+, z] = 0$. To obtain (3) we first apply the procedure at the beginning of the proof of Proposition 2.10 which replaces a clutching function $a(x)z+b(x)$ by $z+[t_0a(x)+b(x)][a(x)+t_0b(x)]^{-1}$ with $0 < t_0 < 1$. In the case $a(x)z + b(x) = \mathbb{1}$ this yields $z + t_0^{-1}\mathbb{1}$. Since $t_0^{-1}\mathbb{1}$ has only the one eigenvalue $t_0^{-1} > 1$, we have $[E_-, \mathbb{1}] = 0$.

Formulas (1) and (3) give $((n+2)E)_- \approx ((n+1)E)_-$, using the fact that the $\pm$-splitting preserves direct sums. So the "minus" summand is independent of n.

Suppose we define

$$\nu(z^{-m}q) = ((n+1)E)_- \otimes (H-1) + E\otimes H^{-m} \in K(X)\otimes K(S^2)$$

for $n \geq deg\, q$. We claim that this is well-defined. We have already noted that "minus" summands are independent of n, so $\nu(z^{-m}q)$ does not depend on n. To see that it is independent of m we must see that it is unchanged when $z^{-m}q$ is replaced by $z^{-m-1}(zq)$. By (2) and (4) we have the first of the following string of equalities:

$$\begin{aligned}
\nu(z^{-m-1}(zq)) &= ((n+1)E)_-\otimes(H-1) + [E,z]\otimes(H-1) + E\otimes H^{-m-1} \\
&= ((n+1)E)_-\otimes(H-1) + E\otimes(H^2-H) + E\otimes H^{-m-1} \\
&= ((n+1)E)_-\otimes(H-1) + E\otimes(H^{-m}-H^{-m-1}) + E\otimes H^{-m-1} \\
&= ((n+1)E)_-\otimes(H-1) + E\otimes H^{-m} \\
&= \nu(z^{-m}q)
\end{aligned}$$

The remaining equalities are obvious except for the step from the second line to the third. By the calculation of the ring $K(S^2)$ in §2.1 we have $(H-1)^2 = 0$, hence $H(H-1) = H-1$, which implies $H^2 - H = H^{-m} - H^{-m-1}$.

The only other choice we have made which might affect the value of $\nu(z^{-m}q)$ is the constant $t_0 < 1$ in the proof of Proposition 2.10. This could be any number sufficiently close to 1, so varying t_0 gives a homotopy of the endomorphism b in Lemma 2.11. This has no effect on the $\pm$-splitting since we can apply Lemma 2.11 to the endomorphism of $E\times I$ given by the homotopy.

We propose to define $\nu([E,f]) = \nu(z^{-m}q)$, so we must see that this depends only on the bundle $[E,f]$, not on the choices of f and $z^{-m}q$. We showed that every bundle over $X\times S^2$ has the form $[E,f]$ for a normalized clutching function f which was unique up to homotopy, and in Proposition 2.8 we showed that Laurent polynomial approximations to homotopic f's are Laurent-polynomial-homotopic. If we apply Propositions 2.9 and 2.10 over $X \times I$ with a Laurent polynomial homotopy as clutching function, we conclude that the two bundles $((n+1)E)_-$ over $X \times \{0\}$ and $X \times \{1\}$ are isomorphic, from which we see that $\nu([E, f])$ is well-defined.

It is easy to check through the definitions to see that ν takes sums to sums, since $L^n(q_1 \oplus q_2) = L^n q_1 \oplus L^n q_2$ and, as previously noted, the $\pm$-splitting in Proposition 2.10 preserves sums. As a result, ν extends to a homomorphism $K(X \times S^2) \to K(X) \times K(S^2)$.

It remains only to verify that $\nu\mu = 1\!\!1$. Since $K(S^2)$ is generated by 1 and H^{-1}, it suffices to show $\nu\mu = 1\!\!1$ on elements $E \otimes H^{-m}$ for $n \geq 0$. We have $\nu\mu(E \otimes H^{-m}) = \nu([E, z^{-m}]) = ((n+1)E)_- \otimes (H - 1) + E \otimes H^{-m} = E \otimes H^{-m}$ since $((n+1)E)_- = 0$, the polynomial q being $1\!\!1$ so that (3) applies.

This completes the proof of Bott Periodicity. $\square$

3. Adams' Hopf Invariant One Theorem

With the hard work of proving Bott Periodicity behind us now, the goal of this section is to prove Adams' theorem on the Hopf invariant, with its famous applications including the nonexistence of division algebras beyond the Cayley octonions:

Theorem 2.12. *The following statements are true only for $n = 1, 2, 4$, and 8:*

(a) $\mathbb{R}^n$ *is a division algebra.*

(b) S^{n-1} *is parallelizable, i.e., there exist $n-1$ tangent vector fields to S^{n-1} which are linearly independent at each point, or in other words, the tangent bundle to S^{n-1} is trivial.*

(c) S^{n-1} *is an H-space.*

To say that S^{n-1} is an H-space means there is a continuous "multiplication" $S^{n-1} \times S^{n-1} \to S^{n-1}$ having a two-sided identity element $e \in S^{n-1}$. This is weaker than being a topological group since associativity and inverses are not assumed. For example, S^1, S^3, and S^7 are H-spaces by restricting the multiplication of complex numbers, quaternions, and Cayley octonions to the respective unit spheres, but only S^1 and S^3 are topological groups since the multiplication of octonions is nonassociative.

The first step in the proof is to reduce (a) and (b) to (c):

Lemma 2.13. *If $\mathbb{R}^n$ is a division algebra, or if S^{n-1} is parallelizable, then S^{n-1} is an H-space.*

Proof: To say that $\mathbb{R}^n$ is a division algebra means that one has a multiplication map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that the maps $x \mapsto ax$ and $x \mapsto xa$ are linear for fixed $a \in \mathbb{R}^n$ and invertible if $a \neq 0$. The latter invertibility condition is equivalent to the multiplication having no zero-divisors. An identity element is not assumed, but the multiplication can be modified to produce an identity in the following way. Choose a unit vector $e \in \mathbb{R}^n$. After composing the multiplication with an invertible linear map $\mathbb{R}^n \to \mathbb{R}^n$ taking e^2 to e we may assume that $e^2 = e$. Let α be the map $x \mapsto xe$ and β the map $x \mapsto ex$. The new product $(x, y) \mapsto \alpha^{-1}(x)\beta^{-1}(y)$ then sends (x, e) to $\alpha^{-1}(x)\beta^{-1}(e) = \alpha^{-1}(x)e = x$, and similarly it sends (e, y) to y.

Having a division algebra structure on $\mathbb{R}^n$ with two-sided identity, an H-space structure on S^{n-1} is given by $(x, y) \mapsto xy/|xy|$, which is well-defined since a division algebra has no zero-divisors.

Now suppose that S^{n-1} is parallelizable, with tangent vector fields $v_1, \cdots, v_{n-1}$ which are linearly independent at each point of S^{n-1}. By the Gram-Schmidt process we may assume the vectors $x, v_1(x), \cdots, v_{n-1}(x)$ are orthonormal for all $x \in S^{n-1}$. We may assume also that at the first standard basis vector e_1, the vectors $v_1(e_1), \cdots, v_{n-1}(e_1)$ are the standard basis vectors $e_2, \cdots, e_n$, by changing the sign of v_{n-1} if necessary to get orientations right, then deforming the vector fields near e_1. Let $\alpha_x \in SO(n)$ send the standard basis to $x, v_1(x), \cdots, v_{n-1}(x)$. Then the map $(x,y) \mapsto \alpha_x(y)$ defines an H-space structure on S^{n-1} with identity element the vector e_1 since α_{e_1} is the identity map and $\alpha_x(e_1) = x$ for all x. ☐

The fact that S^{n-1} is not an H-space when $n-1$ is even and nonzero is a fairly direct application of Bott periodicity. To see this, we observe first of all that the external product $K(S^{2k}) \otimes K(S^{2k}) \to K(S^{2k} \times S^{2k})$ is an isomorphism for $k > 0$ since this is equivalent to the reduced version $\widetilde{K}(S^{2k}) \otimes \widetilde{K}(S^{2k}) \to \widetilde{K}(S^{2k} \wedge S^{2k})$ being an isomorphism, by the same reasoning that showed that the reduced and unreduced versions of Bott periodicity were equivalent. And the reduced external product is an isomorphism since it is an iterate of the reduced Bott periodicity isomorphism.

Thus we have $K(S^{2k} \times S^{2k}) \approx \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$ where α and β are the pullbacks of a generator γ of $\widetilde{K}(S^{2k})$ under the two projections $S^{2k} \times S^{2k} \to S^{2k}$. An additive basis for $K(S^{2k} \times S^{2k})$ is thus $\{1, \alpha, \beta, \alpha\beta\}$. Suppose $\mu : S^{2k} \times S^{2k} \to S^{2k}$ is an H-space multiplication. Then the following argument shows that $\mu^*(\gamma) = \alpha + \beta + m\alpha\beta$ for some integer m. The composition $S^{2k} \xrightarrow{i} S^{2k} \times S^{2k} \xrightarrow{\mu} S^{2k}$ is the identity, where i is the inclusion onto either of the subspaces $S^{2k} \times \{e\}$ or $\{e\} \times S^{2k}$, with e the identity element of the H-space structure. The map i^* for i the inclusion onto the first factor sends α to γ and β to 0, so the coefficient of α in $\mu^*(\gamma)$ must be 1, and similarly the coefficient of β must be 1.

However, the formula $\mu^*(\gamma) = \alpha + \beta + m\alpha\beta$ leads to a contradiction since it implies that $\mu^*(\gamma^2) = (\alpha + \beta + m\alpha\beta)^2 = 2\alpha\beta \neq 0$ but $\gamma^2 = 0$.

There remains the more difficult case that n is even. The first step is a simple construction which associates to a map $g : S^{n-1} \times S^{n-1} \to S^{n-1}$ a map $\widehat{g} : S^{2n-1} \to S^n$. To define this, we regard S^{2n-1} as $\partial(D^n \times D^n) = \partial D^n \times D^n \cup D^n \times \partial D^n$, and S^n we take as the union of two disks D^n_+ and D^n_- with their boundaries identified. Then $\widehat{g}$ is defined on $\partial D^n \times D^n$ by $\widehat{g}(x,y) = |y| g(x, y/|y|) \in D^n_+$ and on $D^n \times \partial D^n$ by $\widehat{g}(x,y) = |x| g(x/|x|, y) \in D^n_-$. Note that $\widehat{g}$ is well-defined and continuous, even when $|x|$ or $|y|$ is zero, and $\widehat{g}$ agrees with g on $S^{n-1} \times S^{n-1}$.

Now we specialize to the case that n is even, or in other words, we replace n by $2n$. For a map $f : S^{4n-1} \to S^{2n}$, let C_f be S^{2n} with a cell e^{4n} attached by f. The quotient C_f / S^{2n} is then S^{4n}, and since $\widetilde{K}^1(S^{4n}) = \widetilde{K}^1(S^{2n}) = 0$, the exact sequence of the pair

(C_f, S^{2n}) becomes a short exact sequence

$$0 \longrightarrow \widetilde{K}(S^{4n}) \longrightarrow \widetilde{K}(C_f) \longrightarrow \widetilde{K}(S^{2n}) \longrightarrow 0$$

Let $\alpha \in \widetilde{K}(C_f)$ be the image of the generator $(H-1)\otimes\cdots\otimes(H-1)$ of $\widetilde{K}(S^{4n})$ and let $\beta \in \widetilde{K}(C_f)$ map to the generator $(H-1)\otimes\cdots\otimes(H-1)$ of $\widetilde{K}(S^{2n})$. The element β^2 maps to 0 in $\widetilde{K}(S^{2n})$ since the square of any element of $\widetilde{K}(S^{2n})$ is zero. So by exactness we have $\beta^2 = h\alpha$ for some integer h. The mod 2 value of h depends only on f, not on the choice of β, since β is unique up to adding an integer multiple of α, and $(\beta+p\alpha)^2 = \beta^2 + 2p\alpha\beta$ since $\alpha^2 = 0$. The value of h mod 2 is called the mod 2 *Hopf invariant* of f. In fact $\alpha\beta = 0$ so h is well-defined in $\mathbb{Z}$ not just $\mathbb{Z}_2$, as we will see in §3.2, but for our present purposes the mod 2 value of h suffices.

Lemma 2.14. *If $g: S^{2n-1}\times S^{2n-1}\to S^{2n-1}$ is an H-space multiplication, then the associated map $\widehat{g}: S^{4n-1}\to S^{2n}$ has Hopf invariant 1.*

Proof: Let $e \in S^{2n-1}$ be the identity element for the H-space multiplication, and let $f = \widehat{g}$. In view of the definition of f it is natural it view the characteristic map Φ of the $4n$-cell of C_f as a map $(D^{2n}\times D^{2n}, \partial(D^{2n}\times D^{2n})) \to (C_f, S^{2n})$. In the following commutative diagram the maps from the third row to the second are induced by Φ. The horizontal maps are the product maps, the diagonal map is external product. To unclutter the diagram we write D^{2n} as D.

$$\begin{tikzcd}
\widetilde{K}(D\times\{e\},\partial D\times\{e\}) & \times & \widetilde{K}(\{e\}\times D,\{e\}\times\partial D) \arrow[dr] & \\
\widetilde{K}(D\times D,\partial D\times D) \arrow[u, "\approx"'] & \times & \widetilde{K}(D\times D, D\times\partial D) \arrow[u, "\approx"'] \arrow[r] & \widetilde{K}(D\times D,\partial(D\times D)) \\
\widetilde{K}(C_f, D^{2n}_-) \arrow[u] \arrow[d, "\approx"] & \times & \widetilde{K}(C_f, D^{2n}_+) \arrow[u] \arrow[d, "\approx"] \arrow[r] & \widetilde{K}(C_f, S^{2n}) \arrow[u, "\approx"'] \arrow[d] \\
\widetilde{K}(C_f) & \times & \widetilde{K}(C_f) \arrow[r] & \widetilde{K}(C_f)
\end{tikzcd}$$

Consider the first column of the diagram. By the definition of an H-space, Φ restricts to a homeomorphism from $D^{2n}\times\{e\}$ to D^{2n}_+, so the element β in the bottom group, which restricts to a generator of $\widetilde{K}(S^{2n})$ by definition, maps to a generator of the top group $\widetilde{K}(D^{2n}\times\{e\},\partial D^{2n}\times\{e\})$. The situation for the second column is similar. The external product map from the top row sends generator cross generator to generator, by Bott periodicity. Therefore by commutativity of the diagram, the product map in the bottom row sends $\beta\times\beta$ to $\pm\alpha$ since α was defined to be the image of a generator of $\widetilde{K}(C_f,S^{2n})$. The equation $\beta^2=\pm\alpha$ says that the Hopf invariant of f is ±1. $\square$

In view of this lemma, Theorem 2.12 becomes a consequence of the following theorem of Adams:

Theorem 2.15. *If $f:S^{4n-1}\to S^{2n}$ is a map whose mod 2 Hopf invariant is 1, then $n=1,\ 2$, or 4.*

The proof of this will occupy the rest of this section.

Adams Operations

The Hopf invariant is defined in terms of the ring structure in K-theory, but in order to prove Adams' theorem, more structure is needed, namely certain ring homomorphisms $\psi^k:K(X)\to K(X)$. Here are their basic properties:

Theorem 2.16. *There exist transformations $\psi^k:K(X)\to K(X)$, defined for all finite cell complexes X and all integers $k\ge 0$, and satisfying:*
(1) $\psi^k f^*=f^*\psi^k$ *for all maps* $f:X\to Y$.
(2) $\psi^k(\alpha+\beta)=\psi^k(\alpha)+\psi^k(\beta)$.
(3) $\psi^k(L)=L^k$ *if L is a line bundle.*
(4) $\psi^k(\alpha\beta)=\psi^k(\alpha)\psi^k(\beta)$.
(5) $\psi^k\circ\psi^\ell=\psi^{k\ell}$.
(6) $\psi^p(\alpha)\equiv\alpha^p \mod p$ *for p prime.*

This last statement means that $\psi^p(\alpha)-\alpha^p=p\beta$ for some $\beta\in K(X)$.

In the special case of a vector bundle E which is a sum of line bundles L_i, properties (2) and (3) give the formula $\psi^k(L_1\oplus\cdots\oplus L_n)=L_1^k+\cdots+L_n^k$. We would like a general definition of $\psi^k(E)$ which specializes to this formula when E is a sum of line bundles. The idea is to use the exterior powers $\lambda^k(E)$. From the corresponding properties for vector spaces we have:

(i) $\lambda^k(E_1\oplus E_2)\approx\bigoplus_i\big(\lambda^i(E_1)\otimes\lambda^{k-i}(E_2)\big)$.
(ii) $\lambda^0(E)=1$, the trivial line bundle.
(iii) $\lambda^1(E)=E$
(iv) $\lambda^k(E)=0$ for k greater than the maximum dimension of the fibers of E.

Recall that we want $\psi^k(E)$ to be $L_1^k+\cdots+L_n^k$ when $E=L_1\oplus\cdots\oplus L_n$ for line bundles L_i. We will show in this case that there is a polynomial s_k with integer coefficients such that $L_1^k + \cdots + L_n^k = s_k(\lambda^1(E), \cdots, \lambda^k(E))$. This will lead us to define $\psi^k(E) = s_k(\lambda^1(E), \cdots, \lambda^k(E))$ for an arbitrary vector bundle E.

To see what the polynomial s_k should be, consider first the auxilliary polynomial $\lambda_t(E) = \sum_i \lambda^i(E)t^i \in K(X)[t]$. This is a finite sum by property (iv), and property (i) says that $\lambda_t(E_1 \oplus E_2) = \lambda_t(E_1)\lambda_t(E_2)$. When $E = L_1 \oplus \cdots \oplus L_n$ this implies that $\lambda_t(E) = \prod_i \lambda_t(L_i)$, which equals $\prod_i(1+L_i t)$ by (ii), (iii), and (iv). The coefficient $\lambda^j(E)$ of t^j in $\lambda_t(E) = \prod_i(1+L_i t)$ is the j^{th} elementary symmetric function σ_j of the L_i's, the sum of all products of j distinct L_i's. To make the discussion completely algebraic, let us introduce the variable t_i for L_i. Thus $(1+t_1)\cdots(1+t_n) = 1 + \sigma_1 + \cdots + \sigma_n$, where σ_j is the j^{th} elementary symmetric polynomial in the t_i's. The fundamental theorem on symmetric polynomials, proved for example in [Lang, p. 134], asserts that every degree k symmetric polynomial in $t_1, \cdots, t_n$ can be expressed as a unique polynomial in $\sigma_1, \cdots, \sigma_k$. In particular, $t_1^k + \cdots + t_n^k$ is a polynomial $s_k(\sigma_1, \cdots, \sigma_k)$, called a *Newton polynomial*. This polynomial s_k is independent of n since we can pass from n to $n-1$ by setting $t_n = 0$. A recursive formula for s_k is:

$$s_k = \sigma_1 s_{k-1} - \sigma_2 s_{k-2} + \cdots + (-1)^{k-2}\sigma_{k-1}s_1 + (-1)^{k-1}k\sigma_k$$

To derive this we may take $n = k$, and then in the identity $(x+t_1)\cdots(x+t_k) = x^k + \sigma_1 x^{k-1} + \cdots + \sigma_k$ substitute $x = -t_i$ to get a formula $t_i^k = \sigma_1 t_i^{k-1} - \cdots + (-1)^{k-1}\sigma_k$. Summing over i then gives the recursion relation. The recursion relation easily yields for example

$$s_1 = \sigma_1, \quad s_2 = \sigma_1^2 - 2\sigma_2, \quad s_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3, \quad s_4 = \sigma_1^4 - 4\sigma_1^2\sigma_2 + 4\sigma_1\sigma_3 + 2\sigma_2^2 - 4\sigma_4.$$

Thus if we define $\psi^k(E) = s_k(\lambda^1(E), \cdots, \lambda^k(E))$, then $\psi^k(E) = L_1^k + \cdots + L_n^k$ when $E = L_1 \oplus \cdots \oplus L_n$, a sum of line bundles.

Verifying that this definition of ψ^k for vector bundles gives operations on $K(X)$ satisfying the properties listed in the theorem will be rather easy if we make use of the following:

The Splitting Principle. *Given a vector bundle $E \to X$ with X a finite cell complex, there is a finite cell complex $F(E)$ and a map $p: F(E) \to X$ such that $p^*: K^*(X) \to K^*(F(E))$ is injective and $p^*(E)$ splits as a sum of line bundles.*

This will be proved later in this section, but for the moment let us assume it and proceed with the proof of Theorem 2.16 and Adams' theorem.

Proof of Theorem 2.16: Property (1) for vector bundles, $f^*(\psi^k(E)) = \psi^k(f^*(E))$, is immediate since $f^*(\lambda^i(E)) = \lambda^i(f^*(E))$. Property (2) for vector bundles, $\psi^k(E_1 \oplus E_2) =$

$\psi^k(E_1)+\psi^k(E_2)$, follows from the splitting principle since we can first pull back to split E_1 then do a further pullback to split E_2, and the preceding formula $\psi^k(L_1\oplus\cdots\oplus L_n)=L_1^k+\cdots+L_n^k$ preserves sums. Since ψ^k is additive on vector bundles, it induces an additive operation on $K(X)$ defined by $\psi^k(E_1-E_2)=\psi^k(E_1)-\psi^k(E_2)$.

For this extended ψ^k the properties (1)-(3) are clear. Property (4) is also easy from the splitting principle: If E is the sum of line bundles L_i and E' is the sum of line bundles L_j', then $E\otimes E'$ is the sum of the line bundles $L_i\otimes L_j'$, so $\psi^k(E\otimes E')=\sum_{i,j}\psi^k(L_i\otimes L_j')=\sum_{i,j}(L_i\otimes L_j')^k=\sum_{i,j}L_i^k\otimes {L_j'}^k=\sum_i L_i^k\sum_j {L_j'}^k=\psi^k(E)\psi^k(E')$. Since (4) holds for vector bundles, it therefore holds for elements of $K(X)$. Similarly, for (5) the splitting principle and (2) reduce us to the case of line bundles, where $\psi^k(\psi^\ell(L))=L^{k\ell}=\psi^{k\ell}(L)$. Likewise, for (6), if $E=L_1+\cdots+L_n$, then $\psi^p(E)=L_1^p+\cdots+L_n^p\equiv(L_1+\cdots+L_n)^p=E^p \mod p$. $\square$

By the naturality property (1), ψ^k restricts to an operation $\psi^k:\widetilde{K}(X)\to\widetilde{K}(X)$ since $\widetilde{K}(X)$ is the kernel of the homomorphism $K(X)\to K(x_0)$ for $x_0\in X$. For the external product $\widetilde{K}(X)\otimes\widetilde{K}(Y)\xrightarrow{\otimes}\widetilde{K}(X\wedge Y)$, we have the formula $\psi^k(\alpha\otimes\beta)=\psi^k(\alpha)\otimes\psi^k(\beta)$ since if one looks back at the definition of $\alpha\otimes\beta$, one sees this was defined as $p_1^*(\alpha)p_2^*(\beta)$, hence

$$\begin{aligned}\psi^k(\alpha\otimes\beta)&=\psi^k(p_1^*(\alpha)p_2^*(\beta))\\&=\psi^k(p_1^*(\alpha))\psi^k(p_2^*(\beta))\\&=p_1^*(\psi^k(\alpha))p_2^*(\psi^k(\beta))\\&=\psi^k(\alpha)\otimes\psi^k(\beta).\end{aligned}$$

This will allow us to compute ψ^k on $\widetilde{K}(S^{2n})\approx\mathbb{Z}$, when ψ^k must be multiplication by some integer since it is an additive homomorphism of $\mathbb{Z}$.

Proposition 2.17. *$\psi^k:\widetilde{K}(S^{2n})\to\widetilde{K}(S^{2n})$ is multiplication by k^n.*

Proof: Consider first the case $n=1$. Since ψ^k is additive, it will suffice to show $\psi^k(\alpha)=k\alpha$ for α a generator of $\widetilde{K}(S^2)$. We can take $\alpha=H-1$ for H the canonical line bundle over $S^2=\mathbb{C}\mathrm{P}^1$. Then

$$\begin{aligned}\psi^k(\alpha)=\psi^k(H-1)&=H^k-1 \qquad \text{by properties (2) and (3)}\\&=(1+\alpha)^k-1\\&=1+k\alpha-1 \qquad \text{since } \alpha^i=(H-1)^i=0 \text{ for } i\ge 2\\&=k\alpha\end{aligned}$$

When $n>1$ we argue by induction using the external product $\widetilde{K}(S^2)\otimes\widetilde{K}(S^{2n-2})\to$

$\widetilde{K}(S^{2n})$, which is an isomorphism. Assuming the desired formula holds in $\widetilde{K}(S^{2n-2})$, we have $\psi^k(\alpha\otimes\beta)=\psi^k(\alpha)\otimes\psi^k(\beta)=k\alpha\otimes k^{n-1}\beta=k^n(\alpha\otimes\beta)$, and we are done. $\square$

Now we can use the operations ψ^2 and ψ^3 and the relation $\psi^2\psi^3=\psi^6=\psi^3\psi^2$ to prove Adams' theorem.

Proof of Theorem 2.15: The definition of the Hopf invariant of a map $f:S^{4n-1}\to S^{2n}$ involved elements $\alpha,\beta\in\widetilde{K}(C_f)$. By Proposition 2.17, $\psi^k(\alpha)=k^{2n}\alpha$ since α is the image of a generator of $\widetilde{K}(S^{4n})$. Similarly, $\psi^k(\beta)=k^n\beta+\mu_k\alpha$ for some $\mu_k\in\mathbb{Z}$. Therefore

$$\psi^k\psi^\ell(\beta)=\psi^k(\ell^n\beta+\mu_\ell\alpha)=k^n\ell^n\beta+(k^{2n}\mu_\ell+\ell^n\mu_k)\alpha$$

Since $\psi^k\psi^\ell=\psi^{k\ell}=\psi^\ell\psi^k$, the coefficient of α in the preceding displayed expression is unchanged when k and ℓ are switched. This gives the relation

$$k^{2n}\mu_\ell+\ell^n\mu_k=\ell^{2n}\mu_k+k^n\mu_\ell,\qquad\text{or}\qquad(k^{2n}-k^n)\mu_\ell=(\ell^{2n}-\ell^n)\mu_k$$

By property (6) of ψ^2, we have $\psi^2(\beta)\equiv\beta^2\bmod 2$. Since $\beta^2=h\alpha$ with h the Hopf invariant of f, the formula $\psi^2(\beta)=2^n\beta+\mu_2\alpha$ implies that $\mu_2\equiv h\bmod 2$, so μ_2 is odd if we assume $h=1$. By the preceding displayed formula we have $(2^{2n}-2^n)\mu_3=(3^{2n}-3^n)\mu_2$, or $2^n(2^n-1)\mu_3=3^n(3^n-1)\mu_2$, so 2^n divides $3^n(3^n-1)\mu_2$. Since 3^n and μ_2 are odd, 2^n must then divide 3^n-1. The proof is completed by the following elementary number theory fact. $\square$

Lemma 2.18. *If 2^n divides 3^n-1 then $n=1,2$, or 4.*

Proof: Write $n=2^\ell m$ with m odd. We shall show that the highest power of 2 dividing 3^n-1 is 2 for $\ell=0$ and $2^{\ell+2}$ for $\ell>0$. This implies the lemma since if 2^n divides 3^n-1, then by this fact, $n\le\ell+2$, hence $2^\ell\le 2^\ell m=n\le\ell+2$, which implies $\ell\le 2$ and $n\le 4$. The cases $n=1,2,3,4$ can then be checked individually.

We find the highest power of 2 dividing 3^n-1 by induction on ℓ. For $\ell=0$, $3^n-1=3^m-1\equiv 2\bmod 4$ since $3\equiv -1\bmod 4$ and m is odd. For $\ell=1$, $3^n-1=3^{2m}-1=(3^m-1)(3^m+1)$. The highest power of 2 dividing the first factor is 2 as we just showed, and the highest power of 2 dividing the second factor is 4 since $3^m+1\equiv 4\bmod 8$ because $3^2\equiv 1\bmod 8$ and m is odd. So the highest power of 2 dividing the product $(3^m-1)(3^m+1)$ is 8. For the inductive step of passing from ℓ to $\ell+1$ with $\ell\ge 1$, i.e., from n to $2n$ with n even, write $3^{2n}-1=(3^n-1)(3^n+1)$. Then $3^n+1\equiv 2\bmod 4$ since n is even, so the highest power of 2 dividing 3^n+1 is 2. Thus the highest power of 2 dividing $3^{2n}-1$ is twice the highest power of 2 dividing 3^n-1. $\square$

The Splitting Principle

The proof of the splitting principal will be easy once we have made an explicit calculation of the ring $K(\mathbb{C}\mathrm{P}^n)$ and proved a general result about the K-theory of fiber bundles called the Leray-Hirsch theorem. This theorem can be used to do quite a few other things as well, as we shall see in the next section.

It will simplify our task slightly if we restrict our attention to finite cell complexes rather than general compact Hausdorff spaces. What we mean by "finite cell complex" is slightly more general than the standard notion of a CW complex with finitely many cells, in that we do not require cells to attach only to cells of lower dimension. The advantage of dropping this requirement is that it then becomes true, and easy to prove, that for a fiber bundle $F \to E \to B$, if F and B are finite cell complexes, so is E. Most of the standard facts about CW complexes are true as well for finite cell complexes, as we explain in the Appendix to this chapter, so for our present purposes the added generality costs nothing.

Proposition 2.19. *If X is a finite cell complex with n cells, then $K^*(X)$ as a group is finitely generated with at most n generators. If all the cells of X are even-dimensional, then $K^1(X) = 0$ and $K^0(X)$ is free abelian with one basis element for each cell.*

Proof: We show this by induction on the number of cells. The complex X is obtained from a subcomplex A by attaching a k-cell, for some k. Then we have an exact sequence

$$\widetilde{K}^*(X/A) \longrightarrow \widetilde{K}^*(X) \longrightarrow \widetilde{K}^*(A)$$

Since $X/A = S^k$, we have $\widetilde{K}^*(X/A) \approx \mathbb{Z}$, so exactness implies that $\widetilde{K}^*(X)$ requires at most one more generator than $\widetilde{K}^*(A)$.

If all cells of X are of even dimension, the exact sequence $0 = K^1(X/A) \to K^1(X) \to K^1(A)$ implies inductively that $K^1(X) = 0$. Then there is a short exact sequence $0 \to \widetilde{K}^0(X/A) \to \widetilde{K}^0(X) \to \widetilde{K}^0(A) \to 0$ with $\widetilde{K}^0(X/A) \approx \mathbb{Z}$. By induction $\widetilde{K}(A)$ is free, so this sequence splits, hence $K^0(X) \approx \mathbb{Z} \oplus K^0(A)$ and the final statement of the proposition follows. $\square$

This proposition applies in particular to $\mathbb{C}\mathrm{P}^n$, which has a cell structure with one cell in each dimenion $0, 2, 4, \cdots, 2n$, so $K^1(\mathbb{C}\mathrm{P}^n) = 0$ and $K^0(\mathbb{C}\mathrm{P}^n) \approx \mathbb{Z}^{n+1}$. The ring structure is given by:

Proposition 2.20. *$K(\mathbb{C}\mathrm{P}^n)$ is the quotient ring $\mathbb{Z}[L]/(L-1)^{n+1}$ where L is the canonical line bundle over $\mathbb{C}\mathrm{P}^n$.*

Thus by the change of variable $x = L - 1$ we see that $K(\mathbb{C}\mathrm{P}^n)$ is the truncated polynomial ring $\mathbb{Z}[x]/(x^{n+1})$.

Proof: The exact sequence for the pair $(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^{n-1})$ gives a short exact sequence

$$0 \longrightarrow K(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^{n-1}) \longrightarrow K(\mathbb{C}\mathrm{P}^n) \xrightarrow{\rho} K(\mathbb{C}\mathrm{P}^{n-1}) \longrightarrow 0$$

We shall prove:

(a_n) $\qquad (L-1)^n$ generates the kernel of the restriction map ρ.

Hence if we assume inductively that $K(\mathbb{C}\mathrm{P}^{n-1}) = \mathbb{Z}[L]/(L-1)^n$, then (a_n) and the preceding exact sequence imply that $\{1, L-1, \cdots, (L-1)^n\}$ is an additive basis for $K(\mathbb{C}\mathrm{P}^n)$. Since $(L-1)^{n+1} = 0$ in $K(\mathbb{C}\mathrm{P}^n)$ by (a_{n+1}), it follows that $K(CP^n) = \mathbb{Z}[L]/(L-1)^{n+1}$, completing the induction.

The rough reason why (a_n) is true is that the kernel of ρ can be identified with $K(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^{n-1}) = \widetilde{K}(S^{2n})$ by the short exact sequence, and Bott periodicity implies that the n-fold reduced external product of the generator $L-1$ of $\widetilde{K}(S^2)$ with itself generates $\widetilde{K}(S^{2n})$. To make a proof out of this we need to relate the external product $\widetilde{K}(S^2) \times \cdots \times \widetilde{K}(S^2) \to \widetilde{K}(S^{2n})$ to the 'internal' product $K(\mathbb{C}\mathrm{P}^n) \times \cdots \times K(\mathbb{C}\mathrm{P}^n) \to K(\mathbb{C}\mathrm{P}^n)$.

According to the usual definition, $\mathbb{C}\mathrm{P}^n$ is the quotient of the unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$ under scalar multiplication. Instead of S^{2n+1} we could equally well take the boundary of the product $D_0^2 \times \cdots \times D_n^2$ where D_i^2 is the unit disk in the i^{th} coordinate of $\mathbb{C}^{n+1}$, where we start the count with $i = 0$ for convenience. Then we have

$$\partial(D_0^2 \times \cdots \times D_n^2) = \textstyle\bigcup_i (D_0^2 \times \cdots \times \partial D_i^2 \times \cdots \times D_n^2)$$

The action of S^1 by scalar multiplication respects this decomposition. The orbit space of $D_0^2 \times \cdots \times \partial D_i^2 \times \cdots \times D_n^2$ under the action is a subspace $C_i \subset \mathbb{C}\mathrm{P}^n$ homeomorphic to the product $D_0^2 \times \cdots \times D_n^2$ with the factor D_i^2 deleted. Thus we have a decomposition $\mathbb{C}\mathrm{P}^n = \bigcup_i C_i$ with each C_i homeomorphic to D^{2n}.

In similar fashion the boundary of $C_0 = D_1^2 \times \cdots \times D_n^2$ is decomposed into n pieces $\partial_i C_0 = D_1^2 \times \cdots \times \partial D_i^2 \times \cdots \times D_n^2$. The inclusions $(C_0, \partial_i C_0) \subset (\mathbb{C}\mathrm{P}^n, C_i)$ give rise to a commutative diagram

$$\begin{tikzcd}
K(D_1^2, \partial D_1^2) \arrow[d, "\approx"] & \times\cdots\times & K(D_n^2, \partial D_n^2) \arrow[d, "\approx"] \arrow[dr] & & \\
K(C_0, \partial_1 C_0) & \times\cdots\times & K(C_0, \partial_n C_0) \arrow[r] & K(C_0, \partial C_0) & \\
K(\mathbb{C}\mathrm{P}^n, C_1) \arrow[u] \arrow[d] & \times\cdots\times & K(\mathbb{C}\mathrm{P}^n, C_n) \arrow[u] \arrow[d] \arrow[r] & K(\mathbb{C}\mathrm{P}^n, C_1 \cup \cdots \cup C_n) \arrow[u, "\approx"'] \arrow[d] \arrow[r, "\approx"] & K(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^{n-1}) \arrow[dl] \\
K(CP^n) & \times\cdots\times & K(\mathbb{C}\mathrm{P}^n) \arrow[r] & K(\mathbb{C}\mathrm{P}^n) &
\end{tikzcd}$$

where the rightward-pointing maps are the n-fold products. The upward vertical map labelled an isomorphism is such because the inclusion $C_0 \hookrightarrow \mathbb{C}\mathrm{P}^n$ induces a homeomorphism $C_0/\partial C_0 \approx \mathbb{C}\mathrm{P}^n/(C_1 \cup \cdots \cup C_n)$. The $\mathbb{C}\mathrm{P}^{n-1}$ at the right side of the diagram sits in $\mathbb{C}\mathrm{P}^n$ in the last n coordinates of $\mathbb{C}^{n+1}$, so is disjoint from C_0, which makes the quotient map $\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^{n-1} \to \mathbb{C}\mathrm{P}^n/(C_1 \cup \cdots \cup C_n)$ a homotopy equivalence.

The element $L-1 \in K(\mathbb{C}\mathrm{P}^n)$ is the image of an element $x_i \in K(\mathbb{C}\mathrm{P}^n, C_i)$ whose image in $K(C_0, \partial_i C_0)$ is a generator. The product of these generators is a generator of $K(C_0, \partial C_0)$. Thus the product $x_1 \cdots x_n$ generates $K(\mathbb{C}\mathrm{P}^n, C_1 \cup \cdots \cup C_n)$, which means that $(L-1)^n$ generates the image of the map $K(\mathbb{C}\mathrm{P}^n, CP^{n-1}) \to K(\mathbb{C}\mathrm{P}^n)$, which equals the kernel of ρ, proving (a_n). □

Now we come to the Leray-Hirsch theorem for K-theory:

Theorem 2.21. *Suppose that $p : E \to B$ is a fiber bundle such that*

(a) *The fiber F and the base B are finite cell complexes, hence also E.*

(b) *$K^*(F)$ is free, and there exist classes $c_1, \cdots, c_k \in K^*(E)$ which restrict to a basis for $K^*(F)$ in each fiber F.*

Then $K^*(E)$, as a module over $K^*(B)$, is free with basis $\{c_1, \cdots, c_k\}$.

Here the $K^*(B)$-module structure on $K^*(E)$ is defined by $\beta \cdot \gamma = p^*(\beta)\gamma$ for $\beta \in K^*(B)$ and $\gamma \in K^*(E)$. Another way to state the conclusion of the theorem is to say that the map $\Phi : K^*(B) \otimes K^*(F) \to K^*(E)$, $\Phi(\sum_i b_i \otimes i^*(c_i)) = \sum_i p^*(b_i)c_i$ for i the inclusion $F \hookrightarrow E$, is an isomorphism.

Proof: This will be by a double induction, first on the dimension of B, then within a given dimension, on the number of cells. The induction starts with the trivial case that B is zero-dimensional, hence a finite discrete set. For the induction step, suppose B is obtained from a subcomplex B' by attaching a cell e^n, and let $E' = p^{-1}(B')$. Then we have a diagram

$$(*)\qquad \begin{array}{ccccccccc} \longrightarrow & K^*(B,B')\otimes K^*(F) & \longrightarrow & K^*(B)\otimes K^*(F) & \longrightarrow & K^*(B')\otimes K^*(F) & \longrightarrow \\ & \big\downarrow{\scriptstyle\Phi} & & \big\downarrow{\scriptstyle\Phi} & & \big\downarrow{\scriptstyle\Phi} & \\ \longrightarrow & K^*(E,E') & \longrightarrow & K^*(E) & \longrightarrow & K^*(E') & \longrightarrow \end{array}$$

where the left-hand Φ is defined by the same formula as above, $\Phi(\sum_i b_i \otimes i^*(c_i)) = \sum_i p^*(b_i)c_i$, but with $p^*(b_i)c_i$ referring now to the relative product $K^*(E,E') \times K^*(E) \to K^*(E,E')$. The right-hand Φ is defined using the restrictions of the c_i's to the subspace

E'. To see that the diagram $(*)$ commutes, we can interpolate between its two rows the row

$$\longrightarrow K^*(E,E')\otimes K^*(F) \longrightarrow K^*(E)\otimes K^*(F) \longrightarrow K^*(E')\otimes K^*(F) \longrightarrow$$

by factoring Φ as the composition $\sum_i b_i \otimes i^*(c_i) \mapsto \sum_i p^*(b_i)\otimes i^*(c_i) \mapsto \sum_i p^*(b_i)c_i$. The upper squares of the enlarged diagram then commute trivially, and the lower squares commute by Proposition 2.5.

By induction on the number of cells of B we may assume the right-hand φ in $(*)$ is an isomorphism. If the left-hand φ is also an isomorphism, then the five-lemma will imply that the middle φ is an isomorphism, finishing the induction step.

Let $\varphi:(D^n,S^{n-1}) \to (B,B')$ be a characteristic map for the attached n-cell. Since D^n is contractible, the pullback bundle $\varphi^*(E)$ is a product, and so we have a commutative diagram

$$\begin{tikzcd}
K^*(B,B')\otimes K^*(F) \arrow[r, "\approx"] \arrow[d, "\Phi"] & K^*(D^n,S^{n-1})\otimes K^*(F) \arrow[d, "\Phi"] \arrow[dr, "\Phi"] & \\
K^*(E,E') \arrow[r, "\approx"] & K^*(\varphi^*(E),\varphi^*(E')) \approx & K^*(D^n\times F, S^{n-1}\times F)
\end{tikzcd}$$

The two horizontal maps are isomorphisms since φ restricts to a homeomorphism on the interior of D^n, hence induces homeomorphisms $B/B' \approx D^n/S^{n-1}$ and $E/E' \approx \varphi^*(E)/\varphi^*(E')$. Thus the diagram reduces us to showing that the right-hand φ, involving the product bundle $D^n\times F \to D^n$, is an isomorphism.

Consider the diagram $(*)$ with (B,B') replaced by (D^n,S^{n-1}). The right-hand Φ in $(*)$ is then an isomorphism by induction on the dimension of B. The middle Φ is an isomorphism by the case of zero-dimensional B since D^n deformation retracts to a point. Therefore by the five-lemma the left-hand Φ in $(*)$ is an isomorphism for $(B,B') = (D^n,S^{n-1})$. $\square$

To deduce the splitting principle from this we consider the following situation. Let $E\to X$ be a vector bundle with fibers $\mathbb{C}^{n+1}$ and compact base X. Then we have an associated projective bundle $p:P(E)\to X$ with fibers $\mathbb{C}\mathrm{P}^n$, where $P(E)$ is the space of lines in E, i.e., one-dimensional linear subspaces of fibers of E. Over $P(E)$ there is the canonical line bundle $L\to P(E)$ consisting of the vectors in the lines of $P(E)$. Via $p^*:K^*(X)\to K^*(P(E))$ we can view $K^*(P(E))$ as a module over $K^*(X)$.

Corollary 2.22. *$K^*(P(E))$ is a free $K^*(X)$-module with basis $1, L-1, \cdots, (L-1)^n$.*

Proof: In view of the calculation of $K^*(\mathbb{C}\mathrm{P}^n)$ in Proposition 2.20, the classes $1, L-1, \cdots, (L-1)^n \in K^*(P(E))$ satisfy the hypothesis of the Leray-Hirsch theorem. $\square$

And finally we have the splitting principle:

Corollary 2.23. *Given a vector bundle $E \to X$ with X a finite cell complex, there is a finite cell complex $F(E)$ and a map $p: F(E) \to X$ such that $p^*: K^*(X) \to K^*(F(E))$ is injective and $p^*(E)$ splits as a sum of line bundles.*

Proof: The preceding corollary says in particular that $p^*: K^*(X) \to K^*(P(E))$ is injective since 1 is among the basis elements. The pullback bundle $p^*(E) \to P(E)$ contains L as a subbundle, hence splits as $L \oplus E'$ for $E' \to P(E)$ the subbundle of $p^*(E)$ orthogonal to L with respect to some choice of inner product. Now repeat the process by forming $P(E')$, splitting off another line bundle from the pullback of E' over $P(E')$. Note that $P(E')$ is the space of pairs of orthogonal lines in fibers of E. After a finite number of repetitions we obtain the flag bundle $F(E) \to X$ described at the end of §1.1, whose points are n-tuples of orthogonal lines in fibers of E, where n is the dimension of E. (If the fibers of E have different dimensions over different components of X, we do the construction for each component separately.) The pullback of E over $F(E)$ splits as a sum of line bundles, and the map $F(E) \to X$ induces an injection on $K^*(X)$ since it is a composition of maps with this property. $\square$

Chapter 3. Characteristic Classes

Characteristic classes are cohomology classes in $H^*(B;R)$ associated to vector bundles $E \to B$ by some general rule which applies to all base spaces B. The four classical types of characteristic classes are:

1. Stiefel-Whitney classes $w_i(E) \in H^i(B;\mathbb{Z}_2)$ for a real vector bundle E.
2. Chern classes $c_i(E) \in H^{2i}(B;\mathbb{Z})$ for a complex vector bundle E.
3. Pontryagin classes $p_i(E) \in H^{4i}(B;\mathbb{Z})$ for a real vector bundle E.
4. The Euler class $e(E) \in H^n(B;\mathbb{Z})$ when E is an orientable n-dimensional real vector bundle.

The Stiefel-Whitney and Chern classes are formally quite similar. Pontryagin classes can be regarded as a refinement of Stiefel-Whitney classes when one takes $\mathbb{Z}$ rather than $\mathbb{Z}_2$ coefficients, and the Euler class is a further refinement in the orientable case. But it is also possible to define all the other classes in terms of Euler classes, so in a sense Euler classes are the most fundamental of all the characteristic classes.

Stiefel-Whitney and Chern classes lend themselves well to axiomatization since in most applications it is the formal properties encoded in the axioms which one uses rather than any particular construction of these classes. The construction we give, using the Leray-Hirsch theorem, has the virtues of simplicity and elegance, and applies to both Stiefel-Whitney and Chern classes simultaneously.

1. Stiefel-Whitney and Chern Classes

Stiefel-Whitney classes are defined for real vector bundles, Chern classes for complex vector bundles. The two cases are quite similar, but for concreteness we shall emphasize the real case, with occasional comments on the minor modifications needed to treat the complex case.

A technical point before we begin: We shall assume without further mention that all base spaces of vector bundles are paracompact, so that the fundamental results of Chapter 1 apply. For the study of characteristic classes this is not an essential restriction since one can always pass to pullbacks over a CW approximation to a given base space, and CW complexes are paracompact.

Axioms and Construction

Here is the main result giving axioms for Stiefel-Whitney classes:

Theorem 3.1. *There is a unique sequence of functions $w_1, w_2, \cdots$ assigning to each real vector bundle $E \to B$ a class $w_i(E) \in H^i(B;\mathbb{Z}_2)$, depending only on the isomorphism type of E, such that:*
(a) $w_i(f^*(E)) = f^*(w_i(E))$ *for a pullback* $f^*(E)$.
(b) $w(E_1 \bigoplus E_2) = w(E_1) \smile w(E_2)$ *for* $w = 1 + w_1 + w_2 + \cdots \in H^*(B;\mathbb{Z}_2)$.
(c) $w_i(E) = 0$ *if* $i > dim\, E$.
(d) *For the canonical line bundle* $E \to \mathbb{R}\mathrm{P}^\infty$, $w_1(E)$ *is a generator of* $H^1(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2)$.

The classes $w_i(E)$ are called *Stiefel-Whitney classes* and the sum $w(E) = 1 + w_1(E) + w_2(E) + \cdots$ is the *total* Stiefel-Whitney class. Note that (c) implies that the sum $1 + w_1(E) + w_2(E) + \cdots$ has only finitely many nonzero terms, so this sum does indeed lie in $H^*(B;\mathbb{Z}_2)$, the direct sum of the groups $H^i(B;\mathbb{Z}_2)$. From the formal identity

$$(1 + w_1 + w_2 + \cdots)(1 + w_1' + w_2' + \cdots) = 1 + (w_1 + w_1') + (w_2 + w_1 w_1' + w_2') + \cdots$$

it follows that the formula $w(E_1 \bigoplus E_2) = w(E_1) \smile w(E_2)$ is just a compact way of writing the relation $w_n(E_1 \bigoplus E_2) = \sum_{i+j=n} w_i(E_1) \smile w_j(E_2)$, where $w_0 = 1$. This relation is sometimes called the *Whitney sum formula*.

The analog for complex vector bundles is:

Theorem 3.2. *There is a unique sequence of functions $c_1, c_2, \cdots$ assigning to each complex vector bundle $E \to B$ a class $c_i(E) \in H^{2i}(B;\mathbb{Z})$, depending only on the isomorphism type of E, such that*
(a) $c_i(f^*(E)) = f^*(c_i(E))$ *for a pullback* $f^*(E)$
(b) $c(E_1 \bigoplus E_2) = c(E_1) \smile c(E_2)$ *for* $c = 1 + c_1 + c_2 + \cdots \in H^*(B;\mathbb{Z})$
(c) $c_i(E) = 0$ *if* $i > dim E$
(d) *For the canonical line bundle* $E \to \mathbb{C}\mathrm{P}^\infty$, $c_1(E)$ *is a chosen generator of* $H^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$.

The classes $c_i(E)$ are called the *Chern classes* of E. As in the real case, the formula in (b) for the total Chern classes can be rewritten in the form $c_n(E_1 \bigoplus E_2) = \sum_{i+j=n} c_i(E_1) \smile c_j(E_2)$, where $c_0 = 1$.

Proof of 3.1 and 3.2: Associated to a vector bundle $\pi : E \to B$ with fiber $\mathbb{R}^n$ is the *projective bundle* $P(\pi) : P(E) \to B$, where $P(E)$ is the space of all lines through the origin in all the fibers of E, and $P(\pi)$ is the natural projection sending each line in $\pi^{-1}(b)$ to $b \in B$. We topologize $P(E)$ as a quotient of the complement of the zero-section of E, the quotient obtained by factoring out scalar multiplication in each fiber. Over a neighborhood U in B where E is a product $U \times \mathbb{R}^n$, this quotient is $U \times \mathbb{R}\mathrm{P}^{n-1}$, so $P(E)$ is a fiber bundle over B with fiber $\mathbb{R}\mathrm{P}^{n-1}$.

We would like to apply the Leray-Hirsch theorem to this bundle $P(E) \to B$, taking cohomology with $\mathbb{Z}_2$ coefficients. To do this we need classes 'c_i'$\in H^i(P(E);\mathbb{Z}_2)$ restricting to generators of $H^i(\mathbb{R}\mathrm{P}^{n-1};\mathbb{Z}_2)$ in each fiber $\mathbb{R}\mathrm{P}^{n-1}$ for $i = 0,\cdots,n-1$. Recall from the proof of Theorem 1.7 that there is a map $g : E \to \mathbb{R}^\infty$ which is a linear injection on each fiber. Projectivizing the map g by deleting zero vectors and then factoring out scalar multiplication produces a map $P(g) : P(E) \to \mathbb{R}\mathrm{P}^\infty$. Let α be a generator of $H^1(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2)$ and let $c = P(g)^*(\alpha) \in H^1(P(E);\mathbb{Z}_2)$. Then the powers c^i for $i = 0,\cdots,n-1$ are the desired classes 'c_i' since a linear injection $\mathbb{R}^n \to \mathbb{R}^\infty$ induces an embedding $\mathbb{R}\mathrm{P}^{n-1} \hookrightarrow \mathbb{R}\mathrm{P}^\infty$ under which α pulls back to a generator of $H^1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{Z}_2)$, hence α^i pulls back to a generater of $H^i(\mathbb{R}\mathrm{P}^{n-1};\mathbb{Z}_2)$. Note that any two linear injections $\mathbb{R}^n \to \mathbb{R}^\infty$ are homotopic through linear injections, so the induced embeddings $\mathbb{R}\mathrm{P}^{n-1} \hookrightarrow \mathbb{R}\mathrm{P}^\infty$ of different fibers of $P(E)$ are all homotopic. Also, we showed in the proof of Theorem 1.7 that any two choices of g are homotopic through maps which are linear injections on fibers, so the classes c^i are independent of the choice of g.

The Leray-Hirsch theorem then says that $H^*(P(E);\mathbb{Z}_2)$ is a free $H^*(B;\mathbb{Z}_2)$-module with basis $1, c, \cdots, c^{n-1}$. Consequently, c^n can be expressed uniquely as a linear combination of these basis elements with coefficients in $H^*(B;\mathbb{Z}_2)$, i.e., there is a unique relation of the form

$$c^n + w_1(E)c^{n-1} + \cdots + w_n(E)\cdot 1 = 0$$

for certain classes $w_i(E) \in H^i(B;\mathbb{Z}_2)$. Recalling the definition of the $H^*(B;\mathbb{Z}_2)$-module structure on $H^*(P(E);\mathbb{Z}_2)$, note that $w_i(E)c^i$ means more precisely $P(\pi)^*(w_i(E)) \smile c^i$. For completeness we define $w_i(E) = 0$ for $i > n$ and $w_0(E) = 1$.

To prove property (a), consider a pullback $f^*(E) = E'$, fitting into a diagram

$$\begin{tikzcd} E' \arrow[r, "\widetilde{f}"] \arrow[d, "\pi'"] & E \arrow[d, "\pi"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

If $g : E \to \mathbb{R}^\infty$ is a linear injection on fibers then so is $g\widetilde{f}$, and it follows that $P(\widetilde{f})^*$ takes the canonical class $c = c(E)$ for $P(E)$ to the canonical class $c(E')$ for $P(E')$. Then

$$\begin{aligned} P(\widetilde{f})^*\Big(\sum_i P(\pi)^*\big(w_i(E)\big) \smile c(E)^{n-i}\Big) &= \sum_i P(\widetilde{f})^* P(\pi)^*\big(w_i(E)\big) \smile P(\widetilde{f})^*\big(c(E)^{n-i}\big) \\ &= \sum_i P(\pi')^* f^*\big(w_i(E)\big) \smile c(E')^{n-i} \end{aligned}$$

so the relation $c(E)^n + w_1(E)c(E)^{n-1} + \cdots + w_n(E)\cdot 1 = 0$ defining $w_i(E)$ pulls back to the relation $c(E')^n + f^*(w_1(E))c(E')^{n-1} + \cdots + f^*(w_n(E))\cdot 1 = 0$ defining $w_i(E')$. By the uniqueness of this relation, $w_i(E') = f^*(w_i(E))$.

Proceeding to property (b), the inclusions of E_1 and E_2 into $E_1 \bigoplus E_2$ give inclusions of $P(E_1)$ and $P(E_2)$ into $P(E_1 \bigoplus E_2)$ with $P(E_1) \cap P(E_2) = \varnothing$. Let $U_1 = P(E_1 \bigoplus E_2) - P(E_1)$ and $U_2 = P(E_1 \bigoplus E_2) - P(E_2)$. These are open sets in $P(E_1 \bigoplus E_2)$ which deformation retract onto $P(E_2)$ and $P(E_1)$, respectively. A map $g: E_1 \bigoplus E_2 \to \mathbb{R}^\infty$ which is a linear injection on fibers restricts to such a map on E_1 and E_2, so the canonical class $c \in H^1(P(E_1 \bigoplus E_2); \mathbb{Z}_2)$ for $E_1 \bigoplus E_2$ restricts to the canonical classes for E_1 and E_2. If E_1 and E_2 have dimensions m and n, consider the classes $\omega_1 = \sum_j w_j(E_1) c^{m-j}$ and $\omega_2 = \sum_j w_j(E_2) c^{n-j}$ in $H^*(P(E_1 \bigoplus E_2); \mathbb{Z}_2)$, with cup product $\omega_1 \omega_2 = \sum_j \big[\sum_{r+s=j} w_r(E_1) w_s(E_2)\big] c^{m+n-j}$. By the definition of the classes $w_j(E_1)$, the class ω_1 restricts to zero in $H^m(P(E_1); \mathbb{Z}_2)$, hence ω_1 pulls back to a class in the relative group $H^m(P(E_1 \bigoplus E_2), P(E_1); \mathbb{Z}_2) \approx H^m(P(E_1 \bigoplus E_2), U_2; \mathbb{Z}_2)$, and similarly for ω_2. The following commutative diagram, with $\mathbb{Z}_2$ coefficients understood, then shows that $\omega_1 \omega_2 = 0$:

$$\begin{array}{ccccc}
H^m(P(E_1 \bigoplus E_2), U_2) & \times & H^n(P(E_1 \bigoplus E_2), U_1) & \xrightarrow{\smile} & H^{m+n}(P(E_1 \bigoplus E_2), U_1 \smile U_2) = 0 \\
\downarrow & & \downarrow & & \downarrow \\
H^m(P(E_1 \bigoplus E_2)) & \times & H^n(P(E_1 \bigoplus E_2)) & \xrightarrow{\smile} & H^{m+n}(P(E_1 \bigoplus E_2))
\end{array}$$

Thus $\omega_1 \omega_2 = \sum_j \big[\sum_{r+s=j} w_r(E_1) w_s(E_2)\big] c^{m+n-j} = 0$ is the defining relation for the Stiefel-Whitney classes of $E_1 \bigoplus E_2$, and so $w_j(E_1 \bigoplus E_2) = \sum_{r+s=j} w_r(E_1) w_s(E_2)$.

Property (c) holds by definition. For (d), recall that the canonical line bundle is $E = \{ (\ell, v) \in \mathbb{R}\mathrm{P}^\infty \times \mathbb{R}^\infty \mid v \in \ell \}$. The map $P(\pi)$ in this case is the identity. The map $g: E \to \mathbb{R}^\infty$ which is a linear injection on fibers can be taken to be $g(\ell, v) = v$. So $P(g)$ is also the identity, hence $c(E)$ is a generator of $H^1(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2)$. The defining relation $c(E) + w_1(E) \cdot 1 = 0$ then says that $w_1(E)$ is a generator of $H^1(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2)$.

The proof of uniqueness of the classes w_i will use a general property of vector bundles called the *splitting principle*:

Proposition 3.3. *For each vector bundle $\pi: E \to B$ there is a space $F(E)$ and a map $p: F(E) \to B$ such that the pullback $p^*(E) \to F(E)$ splits as a direct sum of line bundles, and $p^*: H^*(B; \mathbb{Z}_2) \to H^*(F(E); \mathbb{Z}_2)$ is injective.*

Proof: Consider the pullback $P(\pi)^*(E)$ of E via the map $P(\pi): P(E) \to B$. This pullback contains a natural one-dimensional subbundle $L = \{ (\ell, v) \in P(E) \times E \mid v \in \ell \}$. An inner product on E pulls back to an inner product on the pullback bundle, so we have a splitting of the pullback as a sum $L \bigoplus L^\perp$ with the orthogonal bundle $L^\perp$ having dimension one less than E. As we have seen, the Leray-Hirsch theorem applies to $P(E) \to B$, so

$H^*(P(E);\mathbb{Z}_2)$ is the free $H^*(B;\mathbb{Z}_2)$-module with basis $1, c, \cdots, c^{n-1}$ and in particular the induced map $H^*(B;\mathbb{Z}_2) \to H^*(P(E);\mathbb{Z}_2)$ is injective since one of the basis elements is 1.

This construction can be repeated with $L^\perp \to P(E)$ in place of $E \to B$. After finitely many repetitions we obtain the desired result. ☐

Looking at this construction a little more closely, $L^\perp$ consists of pairs $(\ell, v) \in P(E) \times E$ with $v \perp \ell$. At the next stage we form $P(L^\perp)$, whose points are pairs (ℓ, ℓ') where ℓ and ℓ' are orthogonal lines in E. Continuing in this way, we see that the final base space $F(E)$ is the space of all orthogonal splittings $\ell_1 \oplus \cdots \oplus \ell_n$ of fibers of E as sums of lines, and the vector bundle over $F(E)$ consists of all n-tuples of vectors in these lines. Alternatively, $F(E)$ can be described as the space of all chains $V_1 \subset \cdots \subset V_n$ of linear subspaces of fibers of E with $dim\, V_i = i$. Such chains are called *flags*, and $F(E) \to B$ is the *flag bundle* associated to E. Note that the description of points of $F(E)$ as flags does not depend on a choice of inner product in E.

Now we can finish the proof of Theorem 3.1. Property (d) determines $w_1(E)$ for the canonical line bundle $E \to \mathbb{R}\mathrm{P}^\infty$. Property (c) then determines all the w_i's for this bundle. Since the canonical line bundle is the universal line bundle, property (a) therefore determines the classes w_i for all line bundles. Property (b) extends this to sums of line bundles, and finally the splitting principal implies that the w_i's are determined for all bundles.

For complex vector bundles we can use the same proof, but with $\mathbb{Z}$ coefficients since $H^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \approx \mathbb{Z}[\alpha]$, with α now two-dimensional. The defining relation for the $c_i(E)$'s is modified to be

$$c^n - c_1(E)c^{n-1} + \cdots + (-1)^n c_n(E) \cdot 1 = 0$$

with alternating signs. This is equivalent to changing the sign of α, so it does not affect the proofs of properties (a)-(c), but it has the advantage that the canonical line bundle $E \to \mathbb{C}\mathrm{P}^\infty$ has $c_1(E) = +\alpha$ rather than $-\alpha$ since the defining relation in this case is $c(E) - c_1(E) \cdot 1 = 0$ and $c(E) = \alpha$. ☐

Note that in property (d) for Stiefel-Whitney classes we could just as well use the canonical line bundle over $\mathbb{R}\mathrm{P}^1$ instead of $\mathbb{R}\mathrm{P}^\infty$ since the inclusion $\mathbb{R}\mathrm{P}^1 \hookrightarrow \mathbb{R}\mathrm{P}^\infty$ induces an isomorphism $H^1(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2) \approx H^1(\mathbb{R}\mathrm{P}^1;\mathbb{Z}_2)$. The analogous remark holds for Chern classes as well.

The relation between Stiefel-Whitney and Chern classes is more than formal:

Proposition 3.4. *Regarding an n-dimensional complex vector bundle $E \to B$ as a $2n$-dimensional real vector bundle, then $w_{2i+1}(E) = 0$ and $w_{2i}(E)$ is the image of $c_i(E)$ under the coefficient homomorphism $H^{2i}(B;\mathbb{Z}) \to H^{2i}(B;\mathbb{Z}_2)$.*

For example, since the canonical complex line bundle over $\mathbb{C}\mathrm{P}^\infty$ has c_1 a generator of $H^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$, the same is true for its restriction over $S^2 = \mathbb{C}\mathrm{P}^1$, so by the proposition this 2-dimensional real vector bundle $E \to S^2$ has $w_2(E) \neq 0$.

Proof: The bundle E has two projectivizations $\mathbb{R}\mathrm{P}(E)$ and $\mathbb{C}\mathrm{P}(E)$, consisting of all the real and all the complex lines in fibers of E, respectively. There is a natural projection $p:\mathbb{R}\mathrm{P}(E) \to \mathbb{C}\mathrm{P}(E)$ sending each real line to the complex line containing it, since a real line is all the real scalar multiples of any nonzero vector in it and a complex line is all the complex scalar multiples. This projection p fits into a commutative diagram

$$\begin{array}{ccccc}
\mathbb{R}\mathrm{P}^{2n-1} & \longrightarrow & \mathbb{R}\mathrm{P}(E) & \xrightarrow{\mathbb{R}\mathrm{P}(g)} & \mathbb{R}\mathrm{P}^\infty \\
\downarrow & & \downarrow{\scriptstyle p} & & \downarrow \\
\mathbb{C}\mathrm{P}^{n-1} & \longrightarrow & \mathbb{C}\mathrm{P}(E) & \xrightarrow{\mathbb{C}\mathrm{P}(g)} & \mathbb{C}\mathrm{P}^\infty
\end{array}$$

where the left column is the restriction of p to a fiber of E and the maps $\mathbb{R}\mathrm{P}(g)$ and $\mathbb{C}\mathrm{P}(g)$ are obtained by projectivizing, over $\mathbb{R}$ and $\mathbb{C}$, a map $g:E \to \mathbb{C}^\infty$ which is a $\mathbb{C}$-linear injection on fibers. It is easy to see that all three vertical maps in this diagram are fiber bundles with fiber $\mathbb{R}\mathrm{P}^1$, the real lines in a complex line. The Leray-Hirsch theorem applies to the bundle $\mathbb{R}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty$, with $\mathbb{Z}_2$ coefficients, so for β the standard generator of $H^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$, the $\mathbb{Z}_2$-reduction $\overline{\beta} \in H^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}_2)$ pulls back to a generator of $H^2(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2)$, namely the square α^2 of the generator $\alpha \in H^1(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2)$. Hence the $\mathbb{Z}_2$-reduction $\overline{c}_{\mathbb{C}}(E) = \mathbb{C}\mathrm{P}(g)^*(\overline{\beta}) \in H^2(\mathbb{C}\mathrm{P}(E);\mathbb{Z}_2)$ of the basic class $c_{\mathbb{C}}(E) = \mathbb{C}\mathrm{P}(g)^*(\beta)$ pulls back to the square of the basic class $c_{\mathbb{R}}(E) = \mathbb{R}\mathrm{P}(g)^*(\alpha) \in H^1(\mathbb{R}\mathrm{P}(E);\mathbb{Z}_2)$. Consequently the $\mathbb{Z}_2$-reduction $\overline{c}_{\mathbb{C}}(E)^n + \overline{c}_1(E)\overline{c}_{\mathbb{C}}(E)^{n-1} + \cdots + \overline{c}_n(E)\cdot 1 = 0$ of the defining relation for the Chern classes of E pulls back to the relation $c_{\mathbb{R}}(E)^{2n} + \overline{c}_1(E)c_{\mathbb{R}}(E)^{2n-2} + \cdots + \overline{c}_n(E)\cdot 1 = 0$ which is the defining relation for the Stiefel-Whitney classes of E. This means that $w_{2i+1}(E) = 0$ and $w_{2i}(E) = \overline{c}_i(E)$. $\square$

Examples

Property (a), the naturality of Stiefel-Whitney classes, implies that a product bundle $E = B \times \mathbb{R}^n$ has $w_i(E) = 0$ for $i > 0$ since a product is the pullback of a bundle over a point, which must have $w_i = 0$ for $i > 0$ since a point has trivial cohomology in positive dimensions.

Property (b) implies that taking the direct sum of a bundle with a product bundle does not change its Stiefel-Whitney classes. In this sense Stiefel-Whitney classes are *stable*. For example, the tangent bundle TS^n to S^n is stably trivial since its direct sum with the normal bundle to S^n in $\mathbb{R}^{n+1}$, which is a trivial line bundle, produces a trivial bundle. Hence the Stiefel-Whitney classes $w_i(TS^n)$ are zero for $i > 0$.

From the identity

$$(1 + w_1 + w_2 + \cdots)(1 + w_1' + w_2' + \cdots) = 1 + (w_1 + w_1') + (w_2 + w_1 w_1' + w_2') + \cdots$$

we see that $w(E_1)$ and $w(E_1 \bigoplus E_2)$ determine $w(E_2)$ since the terms on the right side of this formula can be solved successively for w_i' in terms of the w_j's. In particular, if $E_1 \bigoplus E_2$ is the trivial bundle, then $w(E_1)$ determines $w(E_2)$ uniquely.

Let us illustrate this principle by showing that there is no bundle $E \to \mathbb{R}\mathrm{P}^\infty$ whose sum with the canonical line bundle $E_1(\mathbb{R}^\infty)$ is trivial. For we have $w(E_1(\mathbb{R}^\infty)) = 1 + \omega$ where ω is a generator of $H^1(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2)$, and hence $w(E)$ must be $(1+\omega)^{-1} = 1 + \omega + \omega^2 + \cdots$ since we are using $\mathbb{Z}_2$ coefficients. Thus $w_i(E) = \omega^i$, which is nonzero in $H^*(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2)$ for all i. However, this contradicts the fact that $w_i(E) = 0$ for $i > dim E$.

This shows that the compactness assumption in Proposition 1.8 cannot be dropped. To further delineate the question, note that Proposition 1.8 says that the restriction $E_1(\mathbb{R}^{n+1})$ of the canonical line bundle to the subspace $\mathbb{R}\mathrm{P}^n \subset \mathbb{R}\mathrm{P}^\infty$ does have an 'inverse' bundle. In fact, the bundle $E_1^\perp(\mathbb{R}^{n+1})$ consisting of pairs (ℓ, v) where ℓ is a line through the origin in $\mathbb{R}^{n+1}$ and v is a vector orthogonal to ℓ is such an inverse. But for any bundle $E \to \mathbb{R}\mathrm{P}^n$ whose sum with $E_1(\mathbb{R}^{n+1})$ is trivial we must have $w(E) = 1 + \omega + \cdots + \omega^n$, and since $w_n(E) = \omega^n \neq 0$, E must be at least n-dimensional. So we see there is no chance of choosing such bundles E for varying n so that they fit together to form a single bundle over $\mathbb{R}\mathrm{P}^\infty$.

Let us describe now an n-dimensional vector bundle $E \to B$ with $w_i(E) \neq 0$ for all $i \leq n$. This will be the n-fold Cartesian product $(E_1)^n \to (G_1)^n = (\mathbb{R}\mathrm{P}^\infty)^n$ of the canonical line bundle. This vector bundle is the direct sum $\pi_1^*(E_1) \bigoplus \cdots \bigoplus \pi_n^*(E_1)$ where $\pi_i : (G_1)^n \to G_1$ is projection onto the i^{th} factor, so $w((E_1)^n) = \prod_i (1 + \alpha_i) \in \mathbb{Z}_2[\alpha_1, \cdots, \alpha_n] \approx H^*((\mathbb{R}\mathrm{P}^\infty)^n; \mathbb{Z}_2)$. Hence $w_i((E_1)^n)$ is the i^{th} elementary symmetric polynomial σ_i in the α_j's, the sum of all the products of i different α_j's. For example, if $n = 3$ then $\sigma_1 = \alpha_1 + \alpha_2 + \alpha_3$, $\sigma_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3$, and $\sigma_3 = \alpha_1\alpha_2\alpha_3$. Since each σ_i with $i \leq n$ is nonzero in $\mathbb{Z}_2[\alpha_1, \cdots, \alpha_n]$, we have an n-dimensional bundle whose first n Stiefel-Whitney classes are all nonzero.

From naturality it follows that the universal bundle $E_n \to G_n$ must also have all its Stiefel-Whitney classes $w_1(E_n), \cdots, w_n(E_n)$ nonzero. In fact a much stronger statement is true. Let $f:(\mathbb{R}\mathrm{P}^\infty)^n \to G_n$ be the classifying map for $(E_1)^n$ and let $w_i = w_i(E_n)$. The composition

$$\mathbb{Z}_2[w_1,\cdots,w_n] \longrightarrow H^*(G_n;\mathbb{Z}_2) \xrightarrow{f^*} H^*\big((\mathbb{R}\mathrm{P}^\infty)^n;\mathbb{Z}_2\big) \approx \mathbb{Z}_2[\alpha_1,\cdots,\alpha_n]$$

then sends w_i to σ_i. It is a classical algebraic result that the σ_i's are algebraically independent in $\mathbb{Z}_2[\alpha_1,\cdots,\alpha_n]$; see e.g., [Lang, p. 134]. Thus the composition $\mathbb{Z}_2[w_1,\cdots,w_n] \to \mathbb{Z}_2[\alpha_1,\cdots,\alpha_n]$ is injective, hence also the map $\mathbb{Z}_2[w_1,\cdots,w_n] \to H^*(G_n;\mathbb{Z}_2)$. In other words, the classes $w_i(E_n)$ generate a polynomial subalgebra $\mathbb{Z}_2[w_1,\cdots,w_n] \subset H^*(G_n;\mathbb{Z}_2)$.

This subalgebra is in fact equal to $H^*(G_n;\mathbb{Z}_2)$, and the corresponding statement for Chern classes holds as well:

Theorem 3.5. *$H^*(G_n;\mathbb{Z}_2)$ is the polynomial ring $\mathbb{Z}_2[w_1,\cdots,w_n]$ on the Stiefel-Whitney classes $w_i = w_i(E_n)$ of the universal bundle $E_n \to G_n$. Similarly, in the complex case $H^*(G_n(\mathbb{C}^\infty);\mathbb{Z}) \approx \mathbb{Z}[c_1,\cdots,c_n]$ where $c_i = c_i(E_n(\mathbb{C}^\infty))$ for the universal bundle $E_n(\mathbb{C}^\infty) \to G_n(\mathbb{C}^\infty)$.*

The proof we give here for this basic result will be a fairly quick application of the CW structure on G_n constructed at the end of §1.2. A different proof will be given in §3.3 where we also compute the cohomology of G_n with $\mathbb{Z}$ coefficients, which is somewhat more subtle.

Proof: Consider a map $f:(\mathbb{R}\mathrm{P}^\infty)^n \to G_n$ which pulls E_n back to the bundle $(E_1)^n$ considered above. We have noted that the image of f^* contains the symmetric polynomials in $\mathbb{Z}_2[\alpha_1,\cdots,\alpha_n] \approx H^*((\mathbb{R}\mathrm{P}^\infty)^n;\mathbb{Z}_2)$. In fact, the image of f^* is exactly equal to the symmetric polynomials, since if $\pi:(\mathbb{R}\mathrm{P}^\infty)^n \to (\mathbb{R}\mathrm{P}^\infty)^n$ is an arbitrary permutation of the factors, then π pulls $(E_1)^n$ back to itself, so $f\pi \simeq f$, which means that $f^* = \pi^* f^*$, hence the image of f^* is invariant under $\pi^*:H^*((\mathbb{R}\mathrm{P}^\infty)^n;\mathbb{Z}_2) \to H^*((\mathbb{R}\mathrm{P}^\infty)^n;\mathbb{Z}_2)$, and the latter map is just the same permutation of the variables α_i.

To finish the proof in the real case it remains to see that f^* is injective. It suffices to find a CW structure on G_n in which the r-cells are in one-to-one correspondence with monomials $w_1^{r_1}\cdots w_n^{r_n}$ of dimension $r = r_1 + 2r_2 + \cdots + nr_n$, since the number of r-cells in a CW complex X is an upper bound on the dimension of $H^r(X;\mathbb{Z}_2)$ as a $\mathbb{Z}_2$ vector space, and a surjective linear map between finite-dimensional vector spaces is injective if the dimension of the domain is no larger than the dimension of the range.

Monomials $w_1^{r_1}\cdots w_n^{r_n}$ of dimension r correspond to n-tuples $(r_1,\cdots,r_n)$ with $r = r_1 + 2r_2 + \cdots + nr_n$. Such n-tuples in turn correspond to partitions of r into at most n integers, via the correspondence

$$(r_1,\cdots,r_n) \longleftrightarrow r_n \le r_n + r_{n-1} \le \cdots \le r_n + r_{n-1} + \cdots + r_1.$$

Such a partition becomes the sequence $\sigma_1 - 1 \le \sigma_2 - 2 \le \cdots \le \sigma_n - n$, corresponding to the strictly increasing sequence $0 < \sigma_1 < \sigma_2 < \cdots < \sigma_n$. For example, when $n = 3$ we have:

	(r_1, r_2, r_3)	$(\sigma_1 - 1, \sigma_2 - 2, \sigma_3 - 3)$	$(\sigma_1, \sigma_2, \sigma_3)$	dimension
1	0 0 0	0 0 0	1 2 3	0
w_1	1 0 0	0 0 1	1 2 4	1
w_2	0 1 0	0 1 1	1 3 4	2
w_1^2	2 0 0	0 0 2	1 2 5	2
w_3	0 0 1	1 1 1	2 3 4	3
$w_1 w_2$	1 1 0	0 1 2	1 3 5	3
w_1^3	3 0 0	0 0 3	1 2 6	3

The cell structure on G_n constructed in §1.2 has one cell of dimension $(\sigma_1 - 1) + (\sigma_2 - 2) + \cdots + (\sigma_n - n)$ for each increasing sequence $0 < \sigma_1 < \sigma_2 < \cdots < \sigma_n$. So we are done in the real case.

The complex case is entirely similar, keeping in mind that c_i has dimension $2i$ rather than i. The CW structure on $G_n(\mathbb{C}^\infty)$ described in §1.2 also has these extra factors of 2 in the dimensions of its cells, which are all even-dimensional, hence the cellular boundary maps for $G_n(\mathbb{C}^\infty)$ are all trivial and the cohomology with $\mathbb{Z}$ coefficients consists of a $\mathbb{Z}$ summand for each cell. Injectivity of f^* then follows from the algebraic fact that a surjective homomorphism between two free abelian groups of the same finite rank is injective. $\square$

Applications of w_1 and c_1

We saw in §1.1 that the set $Vect^1(X)$ of isomorphism classes of line bundles over X forms a group with respect to tensor product. We know also that $Vect^1(X) = [X, G_1(\mathbb{R}^\infty)]$, and $G_1(\mathbb{R}^\infty)$ is just $\mathbb{R}\mathrm{P}^\infty$, an Eilenberg-MacLane space $K(\mathbb{Z}_2, 1)$. It is a basic fact in algebraic topology that $[X, K(G,n)] \approx H^n(X;G)$ when X has the homotopy type of a CW complex; see Theorem 4.17 of [ATI], for example. Thus one might ask whether the groups $Vect^1(X)$ and $H^1(X;\mathbb{Z}_2)$ are isomorphic. For complex line bundles we have $G_1(\mathbb{C}^\infty) = \mathbb{C}\mathrm{P}^\infty$, a $K(\mathbb{Z}, 2)$, so the corresponding question is whether $Vect^1_{\mathbb{C}}(X)$ is isomorphic to $H^2(X;\mathbb{Z})$.

Proposition 3.6. *The function $w_1 : Vect^1(X) \to H^1(X;\mathbb{Z}_2)$ is a homomorphism, and is an isomorphism if X has the homotopy type of a CW complex. The same is also true for $c_1 : Vect^1_{\mathbb{C}}(X) \to H^2(X;\mathbb{Z})$.*

Proof: The argument is the same in both the real and complex cases, so for definiteness let us describe the complex case. To show that $c_1 : Vect^1_{\mathbb{C}}(X) \to H^2(X)$ is a homomorphism, we first prove that $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$ for the bundle $L_1 \otimes L_2 \to G_1 \times G_1$ where L_1 and L_2 are the pullbacks of the canonical line bundle $L \to G_1 = \mathbb{C}\mathrm{P}^\infty$ under the projections $p_1, p_2 : G_1 \times G_1 \to G_1$ onto the two factors. Since $c_1(L)$ is the generator α of $H^2(\mathbb{C}\mathrm{P}^\infty)$, we know that $H^*(G_1 \times G_1) \approx \mathbb{Z}[\alpha_1, \alpha_2]$ where $\alpha_i = p_i^*(\alpha) = c_1(L_i)$. The inclusion $G_1 \vee G_1 \subset G_1 \times G_1$ induces an isomorphism on H^2, so to compute $c_1(L_1 \otimes L_2)$ it suffices to restrict to $G_1 \vee G_1$. Over the first G_1 the bundle L_2 is the trivial line bundle, so the restriction of $L_1 \otimes L_2$ over this G_1 is $L_1 \otimes 1 \approx L_1$. Similarly, $L_1 \otimes L_2$ restricts to L_2 over the second G_1. So $c_1(L_1 \otimes L_2)$ restricted to $G_1 \vee G_1$ is $\alpha_1 + \alpha_2$ restricted to $G_1 \vee G_1$. Hence $c_1(L_1 \otimes L_2) = \alpha_1 + \alpha_2 = c_1(L_1) + c_1(L_2)$.

The general case of the formula $c_1(E_1 \otimes E_2) = c_1(E_1) + c_1(E_2)$ for line bundles E_1 and E_2 now follows by naturality: We have $E_1 \approx f_1^*(L)$ and $E_2 \approx f_2^*(L)$ for maps $f_1, f_2 : X \to G_1$. For the map $F = (f_1, f_2) : X \to G_1 \times G_1$ we have $F^*(L_i) = f_i^*(L) \approx E_i$, so $c_1(E_1 \otimes E_2) = c_1(F^*(L_1) \otimes F^*(L_2)) = c_1(F^*(L_1 \otimes L_2)) = F^*(c_1(L_1 \otimes L_2)) = F^*(c_1(L_1) + c_1(L_2)) = F^*(c_1(L_1)) + F^*(c_1(L_2)) = c_1(F^*(L_1)) + c_1(F^*(L_2)) = c_1(E_1) + c_1(E_2)$.

As noted above, if X is a CW complex, there is a bijection $[X, \mathbb{C}\mathrm{P}^\infty] \approx H^2(X;\mathbb{Z})$, and the more precise statement is that this bijection is given by the map $[f] \mapsto f^*(u)$ for some class $u \in H^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$. The class u must be a generator, otherwise the map would not always be surjective. Which of the two generators we choose for u is not important, so we may take it to be the class α. The map $[f] \mapsto f^*(\alpha)$ factors as the composition $[X, \mathbb{C}\mathrm{P}^\infty] \to Vect^1_{\mathbb{C}}(X) \to H^2(X;\mathbb{Z})$, $[f] \mapsto f^*(L) \mapsto c_1(f^*(L)) = f^*(c_1(L)) = f^*(\alpha)$. The first map in this composition is a bijection, so since the composition is a bijection, the second map c_1 must be a bijection also. $\square$

The first Stiefel-Whitney class w_1 is closely related to orientability:

Proposition 3.7. *A vector bundle $E \to X$ is orientable iff $w_1(E) = 0$, assuming that X is homotopy equivalent to a CW complex.*

Thus w_1 can be viewed as the obstruction to orientability of vector bundles. An interpretation of the other classes w_i as obstructions will be given in the Appendix to this chapter.

Proof: Without loss we may assume X is a CW complex. By restricting to path-components we may further assume X is connected. There are natural isomorphisms

$$(*) \qquad H^1(X;\mathbb{Z}_2) \xrightarrow{\approx} Hom(H_1(X), \mathbb{Z}_2) \xrightarrow{\approx} Hom(\pi_1(X), \mathbb{Z}_2)$$

from the universal coefficient theorem and the fact that $H_1(X)$ is the abelianization of $\pi_1(X)$. When $X = G_n$ we have $\pi_1(G_n) \approx \mathbb{Z}_2$, and $w_1(E_n) \in H^1(G_n;\mathbb{Z}_2)$ corresponds via $(*)$ to this isomorphism $\pi_1(G_n) \approx \mathbb{Z}_2$ since $w_1(E_n)$ is the unique nontrivial element of $H^1(G_n;\mathbb{Z}_2)$. By naturality of $(*)$ it follows that for any map $f: X \to G_n$, $f^*(w_1(E_n))$ corresponds under $(*)$ to the induced homomorphism $f_*: \pi_1(X) \to \pi_1(G_n) \approx \mathbb{Z}_2$. Thus if we choose f so that $f^*(E_n)$ is a given vector bundle E, we have $w_1(E)$ corresponding under $(*)$ to the induced map $f_*: \pi_1(X) \to \pi_1(G_n) \approx \mathbb{Z}_2$. Hence $w_1(E) = 0$ iff this f_* is trivial, which is exactly the condition for lifting f to the universal cover $\widetilde{G}_n$, i.e., orientability of E. $\square$